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The fully discrete fractional-step method for the Oldroyd model ☆

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A B S T R A C T

In this paper, we consider the stability and convergence results of numerical solutions in fully discrete fractional-step formulation for the Oldroyd model. The proposed numerical method is constructed by the decompositions of the viscosity in time part and the finite element method in space part. With some mild regularity assumptions on the exact solution, the unconditional stability results of the approximate solutions are established. Then, the first order spatial convergence for the "end-of-step" velocity is shown. Based on the above results, the optimal order approximations for the velocity and pressure in L^2-norm are obtained for the mesh size. Finally, two numerical examples are given to illustrate the established theoretical analysis and test the performances of the developed numerical method and show the influences of the memory term for the considered problem.

1. Introduction

In this paper, we consider the following Oldroyd model filling an open bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary Γ and the initial-boundary conditions:

$$\begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p - \int_0^t \rho e^{-\delta(t-s)}\Delta u ds = f, & (x,t) \in \Omega \times (0,T], \\ \nabla\cdot u(x,t) = 0, & (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x), & (x,t) \in \Omega \times \{0\}, \\ u(x,t) = 0, & (x,t) \in \Gamma \times (0,T], \end{cases} \tag{1.1}$$

where $\rho \geq 0$, $1/\delta$, $u = u(x,t)$, $p = p(x,t)$, $f = f(x,t)$, $u_0(x)$ and $T > 0$ represent the viscoelastic coefficient, the relaxation time, the velocity, the pressure, the prescribed external force, the initial velocity, and the final time, respectively.

Problem (1.1) is used as the Oldroyd model because it is the generalization of the initial boundary value problem of the Navier–Stokes equation. For the existence and uniqueness of the solutions of problem (1.1), we can refer to [2,1]. For the research of the numerical methods for problem (1.1), much attention has been attracted in the last decades with several

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numerical methods developed. For example, we can refer to [25,22,26] for the standard Galerkin finite element method (FEM), [38,39] for the penalty FEM, and [36] for the characteristic FEM.

The difficulties encountered in problem (1.1) are mainly of three different kinds: the mixed type of the equations resulted from the coupling of the momentum equation with the incompressibility condition, the treatment of the pressure with a viscous and a convective terms because of the advective–diffusive character of the equations, and finally the nonlinearity of the problem.

The fractional step method is an efficient numerical scheme to overcome above mentioned difficulties. The most attractive feature of the fractional step method is that the original complex problem can be decoupled into several easily solved and linearized subproblems, one only needs to solve some decoupled elliptic equations for different variables at one time step. As a consequence, the computational scale is reduced to save a lot of computational cost. Therefore, the fractional step method has been used to solve the incompressible flow. For instance, we can refer to [3,13,28,32,31] for the Navier–Stokes equations and [29] for natural convection equation.

Another class of the fractional step method, called viscosity-splitting method has also widely been researched. The stability and convergence of the fully discrete version of the so-called θ-method were given by Glowinski in [12] and Fernandez-Cara and Marin Beltran in [10]. The fractional step method and the operator-splitting scheme for the numerical solution of the Navier–Stokes problem were also considered by the well-known predictor–multicorrector algorithm in [8,9]. In this scheme, the time advancement is decomposed into two steps: the first step to solve a linear elliptic problem, while the second step to consider a Stokes problem. Two steps satisfy the full homogeneous Dirichlet boundary condition on the velocity. In [9], the optimal error estimates of $\mathcal{O}(\Delta t)$ in $L^2(H^1)\cap L^\infty(L^2)$ for the end-of-step velocity u^{n+1} and suboptimal bounds of $\mathcal{O}(\Delta t^{\frac{1}{2}})$ in $L^2(L^2)$ for the pressure p^{n+1} were presented. Besides, numerical results of the viscosity-splitting scheme were performed in [6] for illustrating $\mathcal{O}(\Delta t)$ for both velocity and pressure. As a consequence, there exists a gap between the numerical analysis and numerical computations. In [14] the author has obtained the error estimates of $\mathcal{O}(\Delta t)$ in $L^\infty(H^1)$ for the velocity and in $L^2(L^2)$ for the pressure, where a weight at the initial time steps must be included to deduce the optimal error estimates for the pressure. To the best of the authors' knowledge, this might be the most perfect results related to the viscosity-splitting scheme for the Navier–Stokes problem in time-discrete form.

In this paper, a novel fractional step method is considered for the Oldroyd model (1.1). Our proposed numerical method is different from the two-step projection method which is based on the projection of an intermediate velocity field onto the space of solenoidal vector fields. The main difference between our method and the standard projection method is the introduction of a viscous term in the incompressibility step (the second step, see (3.3) and (4.3)), which allows the imposition of the original boundary condition on the end-of-step velocity. The motivations of our study on the fractional-step method are mainly twofold:

- First, it can be used to explain theoretically a class of predictor–multicorrector algorithms widely used in practice (see [6] for a more detailed explanation). These methods are based on an iterative scheme consisting of two steps per iteration with the same structure as the two steps.
- Second, the imposition of the original boundary conditions on the end-of-step velocity. The first step of the method, which is a linear elliptic problem, can be seen as a linearized Burger's problem.

On the other hand, the second step has the structure of the Stokes (mixed) problem whose the discretization leads to a symmetric system of linear equations. Based on the ideas taken from the predictor–multicorrector algorithm, Blasco and Codina developed an iterative technique in [5,6] for the solution of these two problems, in which each iteration consists of the solution of one diagonal linear system and one symmetric, positive (semi)definite system, which is the same for all iterations and time steps (and thus needs being computed and factorized only once at the beginning of the calculations). This iteration shows good convergence results in several test cases, which makes the present fractional-step method feasible from a practical viewpoint.

The main purpose of this paper is to establish the stability and convergence results of the fully discrete fractional step numerical solutions of the Oldroyd model (1.1). We use the time-discrete scheme as an auxiliary problem to simplify the computations of the fully discrete numerical scheme. We adopt the implicit/explicit formulation to treat the first step of splitting scheme because both implicit scheme and explicit scheme for the linear terms and the nonlinear term respectively can result in a linear system with the constant coefficient matrices which can save the computational cost.

The main theoretical results of this paper can be summarized as follows:

1. The unconditional stability results of the fully discrete fractional step numerical solution u_h^{n+1} both in L^2-norm and H^1-norm;
2. The first order convergence of the spatial error of $u^{n+1}-u_h^{n+1}$ in $L^\infty(L^2)\cap L^2(H^1)$;
3. We show that the convergence results for the velocity $E_u^{d,n+1}$ is strongly 2 order in L^2-norm and weakly 2 order in H^1-norm;
4. u_h^{n+1} is strongly second order approximation to $u(t_{n+1})$ in $L^\infty(L^2)$-norm for the mesh size;
5. p_h^{n+1} is strongly first order approximation to $p(t_{n+1})$ in $L^\infty(L^2)$-norm for the mesh size.