Applied Numerical Mathematics 128 (2018) 43–64

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

A stochastic local discontinuous Galerkin method for stochastic two-point boundary-value problems driven by additive noises

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ARTICLE INFO

Article history:
Received 4 October 2017
Received in revised form 31 December 2017
Accepted 28 January 2018
Available online 2 February 2018

Keywords:
Stochastic boundary-value problems
Local discontinuous Galerkin method
White noise
Mean-square convergence
Order of convergence

ABSTRACT

The local discontinuous Galerkin (LDG) method has been successfully applied to deterministic boundary-value problems (BVPs) arising from a wide range of applications. In this paper, we propose a stochastic analogue of the LDG method for stochastic two-point BVPs. We first approximate the white noise process by a piecewise constant random process to obtain an approximate BVP. We show that the solution of the new BVP converges to the solution of the original problem. The new problem is then discretized using the LDG method for deterministic problems. We prove that the solution to the new approximate BVP has better regularity which facilitates the convergence proof for the proposed LDG method. More precisely, we prove L^2 error estimates for the solution and for the auxiliary variable that approximates the first-order derivative. The order of convergence is proved to be two in the mean-square sense, when piecewise polynomials of degree at most p are used. Finally, several numerical examples are provided to illustrate the theoretical results.

1. Introduction and problem statement

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space equipped with a filtration $\{\mathcal{F}_x\}_{a\le x\le b}$ satisfying the usual conditions (that is, it is right continuous and increasing family of sub-σ-algebras of $\mathcal{F}$ while $\mathcal{F}_a$ contains all P-null sets in $\mathcal{F}$). Here, Ω is the space of basic outcomes, $\mathcal{F}$ is the σ-algebra associated with Ω, and P is the (probability) measure on $\mathcal{F}$. Throughout this paper, if X is a random variable defined on a probability space $(\Omega, \mathcal{F}, P)$, we will denote by $\mathrm{E}[X]$ the expected value of X, which is defined by as the Lebesgue integral $\mathrm{E}[X]=\int_\Omega X dP$, provided that the integral exists.

In this paper, we develop and analyze a stochastic local discontinuous Galerkin (SLDG) method for the stochastic two-point boundary-value problem (BVP) driven by an additive white noise, written formally as,

$$-U''+a_1U'+a_0U=F(x)+g(x)\dot{W}(x), \quad x\in(a,b), \quad U(a)=\alpha, \quad U^{(s)}(b)=\beta, \quad s=0,1, \tag{1.1}$$

where the deterministic real-valued functions $F(x)$ and $g(x)$ belong to $L^2(a,b)$. Here, we consider the case of purely Dirichlet boundary conditions ($s=0$) and the case of mixed Dirichlet–Neumann boundary conditions ($s=1$). We assume $a_1\ge 0$ and $a_0\ge 0$ are deterministic constants, α and β are fixed real numbers, and $\dot{W}=\frac{dW}{dx}$ is the white noise on $[a,b]$. Here, $W(x)$ is the one-dimensional real-valued standard Wiener process (or Brownian motion) defined on the probability space $(\Omega, \mathcal{F}, P)$,

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https://doi.org/10.1016/j.apnum.2018.01.023
0168-9274/

whose increment $W(t) - W(s)$ is a Gaussian random variable $N(0, t-s)$ for all $a \le s < t \le b$. In our analysis, we assume that the solution to (1.1) exists and is unique. The existence and uniqueness of the solution to stochastic BVPs (SBVPs) was proved by Nualart and Pardoux in [30,31]. The stochastic BVP (1.1) is the stochastic analogue of the deterministic two-point BVP, which arises in many applications. In our analysis, we always assume that $a_0 \ge$ is a positive number.

It is convenient to restate the above as a system by letting $V = U'$:

$$U' = V, \quad -V' + a_1 V + a_0 U = F(x) + g(x)\dot{W}(x), \quad x \in (a,b), \quad U(a) = \alpha, \quad U^{(s)}(b) = \beta. \tag{1.2}$$

The SBVP (1.2) is, in fact, only a symbolic representation for the integral equations

$$U(x) = U(a) + \int_a^x V(s)ds, \quad x \in (a,b), \tag{1.3a}$$

$$V(x) = V(a) + \int_a^x (a_1 V(s) + a_0 U(s) - F(s))\, ds - \int_a^x g(s) dW(s), \quad x \in (a,b), \tag{1.3b}$$

with the boundary conditions $U(a) = \alpha$ and $U^{(s)}(b) = \beta$.

The integral in (1.3a) and the first integral in (1.3b) are pathwise deterministic Riemann integrals, but the second integral in (1.3b) is an Itô stochastic integral with respect to the Wiener process W. The latter integral cannot be defined pathwise as a deterministic Riemann–Stieltjes integral since the sample paths of the Wiener process, though continuous, are not differentiable or even of bounded variation on any finite interval.

Stochastic differential equations (SDEs) models play a prominent role in a range of application areas when uncertainties or random influences are taken into account. They are used to describe more realistic models. Many areas of applications use SDEs including finance, economics, insurance, chemistry, nonlinear filtering, signal processing and filtering, turbulent flows, fluid flows in random media, neuroscience, several fields of biology and physics, population dynamics and genetics; see *e.g.*, [42,34,33,17,32,24,35] and the references therein. Unfortunately, analytic solutions of most SDEs are not available and numerical techniques are usually required to approximate their solutions. Even for linear SDEs, they are challenging problems in both mathematics and engineering. Designing accurate and efficient numerical methods for SDEs is a recent area of research in computational mathematics and is usually not straightforward. Further research is needed to develop and analyze efficient and robust numerical schemes. For instance, it is well-known that almost all algorithms used for the solution of deterministic ordinary differential equations (ODEs) may not work, or work very poorly for stochastic ODEs since they suffer from poor numerical convergence; see *e.g.*, [24]. We also refer the reader to [21] for a survey of some of numerical methods for SDEs together with a more extensive list of references.

Numerical methods for solving stochastic initial-value problems (IVPs) have been under much study; see, *e.g.*, [24,23,46,1,45,2,5,38,28,37,26,18,36,6,11,10,40,41,44,27,25,12,22,9] and the references therein. However, the theory and numerical solutions of SBVPs have received less attention [3,4,30]. In particular, Arciniega and Allen [3,4] developed shooting methods to numerically solve linear and nonlinear SBVPs. These stochastic shooting methods are analogous to standard shooting methods for numerical solution of deterministic BVPs. However, their convergence can only be guaranteed if the initial point is close enough to the exact value, which is unknown. Allen et al. [2] presented and analyzed finite difference and finite element methods for linear parabolic and elliptic stochastic partial differential equations (SPDEs) driven by white noise. They approximated the white noise processes by piecewise constant random processes to facilitate convergence proofs for the finite element method. They proved that both methods are convergent in the mean-square sense. Their computational experiments suggest that the proposed numerical methods have similar accuracy but the finite element method is computationally more efficient than the finite difference method. In this paper, we use similar approach given in [2] to develop and analyze a stochastic analogue of the local discontinuous Galerkin (LDG) method for (2.1). The LDG method is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. The LDG method was first introduced by Cockburn and Shu in [16] for solving convection–diffusion problems. The main idea of the LDG method is to rewrite a higher-order differential equation into a system of first-order equations and then discretize it by the standard DG method. The LDG method combines many attractive features of the classical finite element and finite volume methods. LDG schemes have been successfully applied to deterministic PDEs arising from a wide range of applications including hyperbolic, elliptic, and parabolic PDEs. Several LDG schemes have been developed for various high order PDEs including the convection–diffusion, wave, and KdV equations. We refer to [43,39] and the references therein for many LDG schemes developed to solve high order PDEs including hyperbolic, parabolic, and elliptic PDEs. Even though LDG methods have been successfully applied to a wide variety of deterministic problems, no LDG method for stochastic two-point BVPs is known to the author. We note that LDG methods for SDEs can be very attractive schemes since it is well-known that LDG methods for deterministic differential equations have good stability properties and high accuracy.

In this work, we propose and analyze a SLDG to solve the SBVP (2.1). The main motivation for the proposed scheme originates from the LDG techniques which have been successfully applied to deterministic differential equations. Motivated by the LDG method for deterministic BVPs, we first construct an approximate BVP on each element by approximating the white noise process using a piecewise constant random process. The new BVP is then discretized using the standard LDG method