Applied Numerical Mathematics 128 (2018) 84–97

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Solving the backward heat conduction problem by homotopy analysis method

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ARTICLE INFO

Article history:
Received 18 November 2016
Received in revised form 13 November 2017
Accepted 2 February 2018
Available online 7 February 2018

Keywords:
Heat conduction
Backward conduction
Ill-posed problem
Homotopy iteration
Convergence
Error bounds
Numerics

ABSTRACT

The backward heat conduction problems aim to determine the temperature distribution in the past time from the present measurement data. For this linear ill-posed problem, we propose a homotopy-based iterative regularizing scheme for noisy input data. The advantages of the proposed scheme are, under general assumptions on the exact initial distribution, we can always ensure the convergence of the homotopy sequence with exact final data as initial guess. For noisy input data, we also establish the error analysis for the regularizing solution with noisy measurement data as our initial guess. Our algorithm is easily implementable with very low computational costs in the sense that we only need to do one iteration from initial guess using the final noisy data directly, while the error is still comparable to other regularizing methods. Numerical implementations are presented.

1. Introduction

The classical heat conduction problems aim to determine the temperature distribution for given initial status and heat source. The simplest heat conduction process in a bounded homogeneous rod can be modeled by the following 1-dimensional initial boundary value problem

$$\begin{cases} u_t = u_{xx} + f(x,t), & (x,t) \in (0,\pi) \times (0,T) := \Omega_T \\ u(0,t) = u(\pi,t) = 0, & t \in [0,T] \\ u(x,0) = g(x), & x \in [0,\pi], \end{cases} \tag{1.1}$$

with $f(x,t)$ being the known heat source and $g(x)$ the initial temperature. Except for the temperature for heat conduction, some other physical quantities governed by the diffusion process such as the concentration of the pollutants can also be described by this well-posed mathematical model.

However, in many engineering areas such as archeology and environgeology, we need to consider the so-called backward problems, namely, instead of giving the initial status $g(x)$, the distribution or its noisy measurement of physical quantity at some final time $T > 0$ is specified, we are also asked to find $u(x,t)$ for $0 \le t < T$. Physically, such kinds of time-reverse problems try to infer the status before from its current information. Since this process conflicts with the physical principle

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https://doi.org/10.1016/j.apnum.2018.02.002

of entropy decreasing, such problems are of great importance and consequently receive much attentions in engineering communities, see [10,17,19,25]. Mathematically, they belong to the category of ill-posed problems for PDEs, see the researches under general framework with regularizing schemes [5,9,18].

Let us state this problem precisely. Assume that $u[g](x,T) := \phi(x)$ is the exact field at $t = T$ generated by (1.1) from some unknown initial distribution $g \in L^2(0,\pi)$. Now we are given the noisy data of $\phi(x)$, denoted by $\phi^\delta(x)$, satisfying

$$\|\phi^\delta - \phi\|_{L^2(0,\pi)} \le \delta. \tag{1.2}$$

Then the backward heat conduction problem is to reconstruct the initial data $g(x)$, or, consequently, $u(x,t)$ for $0 \le t < T$ in terms of (1.1)–(1.2) from giving noisy data $\phi^\delta(x)$. In the sequel, we always assume that the source $f(x,t)$ is known.

It is well-known that, although $\phi(x) = u[g](x,T)$, $\phi^\delta(x)$ may not be the final temperature yielded from any initial data $u(x,0)$ even if for arbitrarily small δ. This fact means we cannot approximate $u(x,t)$ by solving (1.1) directly with initial data $u(x,0) = g(x)$ replaced by final data $u(x,T) = \phi^\delta(x)$. Due to the great importance of such kinds of problems, the backward diffusion problems have been studied for a long time with extensive literatures, for example, see [2,6,7,11,22]. The main research topics are the conditional stability for stable computations and the constructions of the regularizing solution for noisy input data which can approximate the exact physical quantity. Roughly speaking, the error bounds of the approximations depend both on the regularizing schemes and also on the *a-priori* assumptions on the exact initial value $g(x)$. The main arguments dealing with these problems are the quasi-reverse methods [3,5,9], the integral equations methods [4,8] as well as the optimization schemes [15]. The general schemes are firstly to reconstruct the initial distribution $u(x,0) = g(x)$ from $\phi^\delta(x)$ and then to determine $u(x,t)$ for $0 < t < T$ by solving the well-posed forward problem (1.1). However, to get a satisfactory approximation to $g(x)$ from the noisy data ϕ^δ, the strict *a-priori* restrictions on $g(x)$ should be assumed. The backward problems for abnormal diffusion process with time fractional derivatives in the governed equations have also been considered recently, see [16,26,29], for example.

In recent years, the homotopy analysis methods (HAM) have been proposed to solve some PDEs problems by iteration process, see [1,13,20,21,23,28], showing powerful effects on solving many engineering problems. Although several theoretical issues have been considered [14,27], the rigorous mathematical analysis such as the convergence and the error analysis seems still be in the initial stages for this algorithm. On the other hands, it has been found that the HAM schemes can also be applied to solve some inverse problems, see [21,24,28].

The concept of homotopy is originated from the topology. A simple application is to find the zero point x^* of a function $r(x)$ by considering the homotopy equation $\phi(x;p) = 0$ for $p \in [0,1]$, where the homotopy function $\phi(x,p)$ is introduced to map $\phi(x,0) = r_0(x), \phi(x,1) = r(x)$ for some simple known function $r_0(x)$. The homotopy scheme aims to determine x^* from the known root x_0 of $r_0(x) = 0$ by some iteration process based on the power series expansion of $\phi(x,p)$ in terms of $p \in [0,1]$. The main difficulty for the successful implementations of the homotopy scheme is the convergence of the iteration solution at $p = 1$, which ensures that the iterative solution converges indeed to the root of $r(x) = 0$, see [27] for some tutorials. For homotopy analysis method applied in solving some ODEs/PDEs problems, the theoretical analysis is still in the initial stage, since the convergence of the homotopic Maclaulin series at $p = 1$ heavily depends both on the functional describing the problem and on the artificially introduced homotopy transform. For more details, see [12,14].

In this paper, we propose an iteration scheme based on the homotopy iteration to determine $u(x,t)$ in $0 \le t < T$ directly from the final noisy data $\phi^\delta(x)$, without the necessity to reconstruct $g(x)$ firstly. Our iteration process begins with the given final measurement as initial guess and the convergence is ensured. The approximation error of this iteration process for noisy input data is also analyzed rigorously.

We initialize the application of the homotopy scheme to solve the backward diffusion problems by considering heat conduction problem (1.1)–(1.2). Different from the homotopy schemes for solving well-posed ODE/PDE problems, our problem is ill-posed. That is, we need to consider both the convergence property of the iteration process and the error propagation during the iteration process. Due to the ill-posedness, the iteration should stop at some reasonable step depending on the noisy level to avoid the amplification of error. Once the number of series truncations is determined in terms of our error estimates, our iterative scheme needs only one iteration essentially, which provides a very efficient computational algorithm.

There exist extensive literatures related to the backward problems. The backward problems for general abstract equation $u'(t) + \mathbb{A}u(t) = 0$ are considered in [5,9,18] with convergence analysis of the regularizing solution, but without implementable numerical schemes. Especially, [9] establishes the regularizing theory in the Banach space. In [10], the 2-dimensional backward heat conduction problem in a bounded domain of arbitrary shape is considered, with a higher order difference scheme based on the Fourier transform and Taylor expansion, to overcome the difficulty of ill-posedness. The well-known meshless techniques based on the fundamental solution to standard heat equation $u_t(x,t) = \Delta u(x,t)$ in $x \in \Omega \subset \mathbb{R}^d$ are numerically studied for backward problem in [19] for $d = 1,2$ and in [25] for $d = 3$. Although this scheme has shown its validity for many engineering problems, the theoretical analysis on the locations of sources are still absent. In [17], the authors solved the backward heat conduction problems by the backward time dynamics with some numerical implementations for 1-dimensional heat conduction problem. Compared with all the above existing works, our homotopy reconstruction algorithm proposed in this paper is an explicit iteration scheme with rigorous convergence analysis and error estimates on the regularizing solution. Moreover, the theoretical framework established here can be generalized to consider other linear ill-posed problems, not necessarily the backward heat conduction problems.

This paper is organized as follows. In section 2, we establish the iteration scheme by the homotopy analysis for (1.1)–(1.2), with the convergence analysis on this process for exact final data $\phi(x)$. Then in section 3, we apply this iteration process to