Applied Numerical Mathematics 127 (2018) 142–163

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Decoupled, semi-implicit scheme for a coupled system arising in magnetohydrodynamics problem

Yuan Li *, Yanjie Ma, Rong An

College of Mathematics and Information Science, Wenzhou University, Wenzhou 325025, PR China

ARTICLE INFO

Article history:
Received 6 March 2017
Received in revised form 31 December 2017
Accepted 7 January 2018
Available online 10 January 2018

Keywords:
Magnetohydrodynamics equations
Semi-implicit scheme
Finite element approximation
Temporal-spatial error estimates

ABSTRACT

In this paper, we focus on a decoupled, linearized semi-implicit Galerkin FEM scheme for a MHD system coupled by the time-dependent Navier–Stokes equation with the steady Maxwell's equations in three-dimensional convex domain. First, additional regularities of the solution to the coupled MHD system are derived. By using $\mathbf{H}^1$-conforming finite element to approximate the magnetic field, it is shown that the proposed semi-linearized scheme is of the first-order convergence order of the velocity field, the magnetic field and the pressure under the time step condition $\Delta t = \mathcal{O}(h)$.

1. Introduction

The incompressible magnetohydrodynamics (MHD) equations are used to describe the flow of a viscous, incompressible and electrically conducting fluid. The governing equations form a multifield problem which arises in several applications such as liquid metals in magnetic pumps or aluminum electrolysis. For comprehensive accounts of the physical background of the MHD equations, we refer to Hughes [14] and Moreau [16].

It is well-known that the MHD equations consist of a coupling between the Navier–Stokes equations of continuum fluid mechanics and the Maxwell equations of electromagnetism. There have several papers which are devoted to the design and the analysis of numerical schemes for the simulation of the stationary or nonstationary MHD problem. For the stationary MHD problem which is coupled by the stationary Navier–Stokes equations and the stationary Maxwell equations, Gunzburger, Meir & Peterson in [9] gave a detailed existence theory and convergence analysis of finite element method in a bounded and simply-connected domain $\Omega \subset \mathbf{R}^3$ that was either convex or whose boundary was of class $C^{1,1}$ by using $\mathbf{H}^1$-conforming finite element to approximate the magnetic field. A stabilized mixed finite element method was developed by Gerbeau [5]. For the non-convex domain or Lipschitz polyhedra domain of engineering practice, the magnetic field may have regularity below $\mathbf{H}^1(\Omega)$. In this case, the $\mathbf{H}^1$-conforming finite element approximation solution for the magnetic field, albeit stable, may not converge to the correct magnetic field. A mixed finite element formulation based on $\mathbf{H}(\text{curl})$-element for the magnetic field was proposed and studied by Schötzau in [18] for the stationary MHD problem. For the nonstationary MHD problem which was coupled by the transient Navier–Stokes equations and the transient Maxwell equations, He proposed a linearized semi-implicit Euler scheme in [10], where $\mathbf{L}^2$-unconditional convergence of this scheme was proved by using the negative norm technique. A fully discrete Crank–Nicolson scheme was studied in [22]. We notice that a decoupled numerical scheme was studied by Zhang & He [21]. The difference with this paper is that they consider the nonstation-

* Corresponding author.
E-mail addresses: math.liyuan@gmail.com, liyuan@wzu.edu.cn (Y. Li).

https://doi.org/10.1016/j.apnum.2018.01.005
0168-9274/

ary MHD problem in [21]. Recently, Xu and his collaborators designed and analyzed a structure preserving finite element scheme in [13,15], where the main feature was that the proposed finite element scheme naturally preserved the important Gauss's law. Some decoupled, unconditionally stable schemes were studied in [2,11], where the main feature was that the nonstationary MHD problem was rewritten in Elsässer variables [20] instead of primitive variables. Then the MHD problem can be decoupled in an unconditionally stable way into two Oseen problems at every time step. There have some other fully discrete schemes, such as the projection methods [3,17], the fractional-step method [4]. For a review of numerical methods for the MHD problem, we refer readers to Gerbeau, Bris & Lelièvre [8].

In this paper, we consider the following hybrid MHD system which is coupled by the transient Navier–Stokes equations and the stationary Maxwell equations in the primitive variable formulation:

$$\frac{\partial \mathbf{u}}{\partial t} - \frac{1}{Re}\Delta \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p + S\widetilde{\mathbf{b}} \times \mathrm{curl}\,\widetilde{\mathbf{b}} = \mathbf{f} \quad \text{in } \Omega \times (0,T], \tag{1.1}$$

$$\mathrm{div}\,\mathbf{u} = 0 \quad \text{in } \Omega \times (0,T], \tag{1.2}$$

$$\frac{1}{Rm}\mathrm{curl}\,(\mathrm{curl}\,\widetilde{\mathbf{b}}) - \mathrm{curl}\,(\mathbf{u} \times \widetilde{\mathbf{b}}) = 0 \quad \text{in } \Omega \times (0,T], \tag{1.3}$$

$$\mathrm{div}\,\widetilde{\mathbf{b}} = 0 \quad \text{in } \Omega \times (0,T], \tag{1.4}$$

$$\mathbf{u}(x,0) = \mathbf{u}_0 \quad \text{in } \Omega, \tag{1.5}$$

$$\mathbf{u} = 0, \quad \widetilde{\mathbf{b}}\cdot\mathbf{n} = q, \quad \mathrm{curl}\,\widetilde{\mathbf{b}} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \times [0,T], \tag{1.6}$$

with some $T > 0$, where $\Omega \subset \mathbf{R}^3$ is a bounded, simply-connected and convex domain. Positive constants Re, Rm and S denote the Reynolds number, the magnetic Reynolds number and the coupling number, respectively. The vector-value function $\mathbf{f}$ represents the body forces applied to the fluid. $\mathbf{n}$ denotes the unit outward normal vector on $\partial\Omega$. The initial vector functions $\mathbf{u}_0$ satisfies the compatibility condition div $\mathbf{u}_0 = 0$. The unknowns in (1.1)–(1.6) are the fluid velocity field $\mathbf{u}$, the pressure p and the magnetic field $\mathbf{b}$. Such a system arises for instance in the modelling of an industrial process when the magnetic phenomena is known to reach their steady state "infinitely" faster than the hydrodynamics phenomena. From (1.3) we can see that the equations related to the magnetic field are elliptic. Moreover, the ellipticity of the equations depends on the velocity field $\mathbf{u}$. Briefly speaking, if the velocity field $\mathbf{u}$ becomes too large, the MHD system (1.1)–(1.6) may become ill-posed. Under restrictive assumptions depending upon the physical data, Gerbeau & Bris in [6,7] showed that the MHD system (1.1)–(1.6) existed a unique local strong solution on a time interval $[0, T^\star]$ for some $T^\star < T$.

As described above, although there have many works about numerical schemes for the stationary or nonstationary MHD problems, but no numerical methods are studied for the hybrid MHD system (1.1)–(1.6). In this paper, we will present a first-order decoupled, semi-implicit FEM scheme for numerical simulation of (1.1)–(1.6). The proposed fully discrete FEM scheme consists of two steps. At first step, we solve the magnetic field by using semi-implicit scheme for the nonlinear term curl $(\mathbf{u} \times \mathbf{b})$. At second step, we solve the velocity field and the pressure by using semi-implicit scheme for the nonlinear term $(\mathbf{u}\cdot\nabla)\mathbf{u}$. Because the stability, the convergence and the convergence rates are important in studying the fully discrete FEM schemes for the nonlinear parabolic problems, in this paper, based on the mixed finite element which consists of approximating the velocity field and the pressure with Taylor–Hood element, the magnetic field with the classical $\mathbf{H}^1$-conforming Lagrange finite element, we will show the optimal temporal-spatial error estimates of the velocity field, the pressure and the magnetic field under the time step condition $\Delta t = \mathcal{O}(h)$.

This paper is organized as follows. In next section, we begin with some notations and lay out the proofs of some additional regularities of the solution to (1.1)–(1.6). These regularities are used in the proof of the temporal-spatial error estimates. The proposed decoupled, semi-implicit finite element scheme and the regularities of the time discrete schemes are presented in Section 3. Especially, for the time discrete schemes, inspired by [6,7], we can show that a unique solution exists under restrictive assumptions depending upon the physical data. Optimal temporal-spatial error estimates are given in Section 4. Meanwhile, we show these error estimates by using mathematical induction method under some reasonable regularities. The numerical results are given to verify our theoretical analysis in Section 5. Final section is a summary section about the main results derived in this paper.

2. Mathematical setting and some regularity results

Throughout this paper, the symbols $C, C_0, C_1, \cdots$ are used to denote the general positive constants being independent of the time step Δt and the mesh size h. For the mathematical setting of the MHD system (1.1)–(1.4) with the initial and boundary conditions (1.5)–(1.6), we introduce some function spaces and their associated norms. For $k \in \mathbb{N}^+$ and $1 \le p \le +\infty$, let $W^{k,p}(\Omega)$ denote the standard Sobolev spaces. The norms in $W^{k,p}(\Omega)$ are defined by

$$||u||_{W^{k,p}} = \begin{cases} \left(\sum_{|\beta| \le k} \int_\Omega |\nabla^\beta u|^p dx \right)^{1/p} & \text{for } 1 \le p < +\infty, \\ \sum_{|\beta| \le k} \sup_\Omega |\nabla^\beta u| & \text{for } p = +\infty, \end{cases}$$