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# An iteratively adaptive multiscale finite element method for elliptic interface problems

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ABSTRACT

We develop and study a framework of multiscale finite element method (MsFEM) for solving the elliptic interface problems. Finding an appropriate boundary condition setting for local multiscale basis function problems is the current topic in the MsFEM research. In the proposed framework, which we call the iteratively adaptive MsFEM (i-ApMsFEM), the local-global information exchanges through iteratively updating the local boundary condition. Once the multiscale solution is recovered from the solution of global numerical formulation on coarse grids, which couples these multiscale basis functions, it provides feedback for updating the local boundary conditions on each coarse element. The key step of i-ApMsFEM is to perform a few smoothing iterations for the multiscale solution to eliminate the high-frequency error introduced by the inaccurate coarse solution before it is used for setting the boundary condition. As the method iterates, the quality of the MsFEM solution improves, since these adaptive basis functions are expected to capture the multiscale feature of the approximate solution more accurately. We demonstrate the advantage of the proposed method through some numerical examples for elliptic interface benchmark problems.



## 1. Introduction

Multiscale finite element/finite volume methods [7,8,14,15,18,19] have been demonstrated to be accurate and efficient numerical schemes for solving physical and engineering problems modeled by partial differential equations that exhibit multiscale phenomena. Contrary to classical Galerkin finite element methods using polynomial-type basis functions, typically lower order ones, such as linear or quadratic functions, the key ingredient of multiscale finite elements is the set of basis functions constructed locally by solving original partial differential equations restricted to the so-called *coarse element* with some proper boundary conditions. The multiscale feature of the solution is taken account through these basis functions in the global coarse formulation so that it provides a more accurate approximate solution when using much fewer grid points than the classical approaches.

It is known in the multiscale finite element literature that the choice of the boundary condition for the construction of multiscale basis functions plays an important role in the overall performance of the multiscale method. Generally speaking, an optimal boundary condition is often not known in *a priori* and is problem-dependent. Depending on the settings of the boundary conditions for the multiscale basis functions, the multiscale finite element methods can typically be classified

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into the following categories. The first one is to employ the local information purely. The simplest one is to use a linear boundary condition that connects two adjacent coarse nodal points, e.g., the residual-free bubble [9,11,12]. A more sophisticated choice is to employ the solution of the original differential operator restricted to each edge of the coarse elements [13,18]. However, if the selection of the boundary conditions is not able to reflect the nature of multiscale solution behavior, the error introduced by the boundary condition will cause a large error in the global coarse solution. In [18,19] the two-scale analysis based on the homogenization theory revealed that the convergence rate of the multiscale method degrades mainly due to boundary resonance error between the coarse element size and the characteristic length. The problem becomes severe for the case that these two quantities are close to each other. One remedy solution is to use the oversampling technique [18,19], where the multiscale basis functions are constructed by using the information on the overlapped subdomains in order to reduce the boundary effects. The second class is to employ the global solution information. One can solve a simplified model problem and use its solution on whole domain for setting the boundary condition for multiscale basis problem. For example, in the two-phase flow simulation, Efendiev et al. [7] demonstrate the advantage of using the projection of the global fine-scale solution of the single-phase equation at the initial time for setting the boundary condition numerically.

Interface problems arise from a variety of applications in computational science and engineering [3,5,21]. For the elliptic interface problems, the convergence rate of the standard Galerkin finite element method is shown, either numerically or theoretically, to be suboptimal [23] if the finite element nodal points cannot resolve the interface curve. This is often the case in many practical applications, and a body-fitting mesh is usually expensive to construct. In this paper, we study a new framework of MsFEM named "iteratively adaptive multiscale finite element methods" (or i-ApMsFEM) for solving elliptic interface problems [4,5,20,23]. In the proposed framework, the local-global information iteratively exchanges through the updated local boundary condition for these multiscale basis problems. Once the multiscale solution is recovered from the solution of the global numerical formulation on coarse grids, which couples these multiscale basis functions, it provides feedback for updating the local boundary conditions on each coarse element. As we increase the number of iterations, the quality of MsFEM solutions improves, since these adaptive basis functions are expected to be able to capture the multiscale feature of the approximate solution more accurately. The key step of i-ApMsFEM is to perform a few steps of smoothing iterations for the multiscale solution before it is used for setting the boundary condition for the multiscale basis functions in order to eliminate the high-frequency error introduced by the inaccurate coarse solution. Alternatively, the other class of iterative multiscale finite volume methods was recently developed by Jenny and his coworkers [14,15]. It can be viewed as the classical multiscale finite element space enriched by a space spanned by adaptive bubble-shaped basis functions, in which both of multiscale basis functions and bubble-shaped basis functions are defined on the dual coarse cells. Then the multiscale solution is recovered on the primary coarse grid. Unlike residual-free bubble functions requiring the zero values on all boundaries, the bubble type basis functions in this method are iteratively updated. The bubble functions along the boundary are allowed to take nonzero values depending on the global solution. Only the values of the bubble functions at the nodal points are kept to be zero. Other similar classes of iterative MsFEMs were proposed by Millward [23]. It can be viewed as a generalization of oversampled MsFEM, in which the boundary values for the local multiscale basis functions are adaptively updated via the data exchange on the overlapping regions along the boundary curve.

The rest of the paper is organized as follows. In Section 2, we describe a general framework of the proposed iteratively adaptive finite element methods for solving the second order PDEs. In Section 3, we present the numerical results for the high-contrast interface problem, and we give some concluding remarks in Section 4.

## 2. A description of i-ApMsFEM method for a PDE model problem

### 2.1. Model problem

Let $\Omega$ be an open and bounded domain with $\partial\Omega$ in $\mathbb{R}^2$. For simplicity, we assume that $\partial\Omega$ is a polygonal curve, in which case we say that $\Omega$ is a polygonal domain. (If $\partial\Omega$ is a curve, we can approximate it with a polygon.) Consider the following model problem,

$$\begin{cases} \mathcal{L}u = f \\ u = g, \end{cases} \tag{1}$$

where $\mathcal{L}$ is a second-order linear elliptic PDE operator with a discontinuous coefficient, $u$ is an unknown scalar function, $g$ is a prescribed Dirichlet-type datum, and $f$ is a given source function. We also assume that this problem is well posed.

### 2.2. Adaptive multiscale finite element formulation

Next, we describe i-ApMsFEM in detail as follows. As shown in Fig. 1, consider the quadrilateral coarse mesh $\mathcal{T}^H = \{K^c\}$ with an element size $H$. For each $K^c$, we further partition it into fine meshes $\mathcal{T}^h = \{K\}$ with an element size $h$ for the purpose of the numerical construction of multiscale finite element basis and bubble functions on the coarse elements. For simplicity, we assume the interfacial nodal points between neighboring coarse elements match each other. Let $s$ be the