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Numerical solution of singularly perturbed linear parabolic system with discontinuous source term

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A B S T R A C T

In this work, we consider a parabolic system with an arbitrary number of linear singularly perturbed equations of reaction–diffusion type coupled in the reaction terms with a discontinuous source term. The diffusion term in each equation is multiplied by a small positive parameter, but these parameters may have different order of magnitude. The components of the solution have boundary and interior layers that overlap and interact. To obtain the approximate solution of the problem we construct a numerical method by combining the backward-Euler method on an uniform mesh in time direction, together with a central difference scheme on a variant of piecewise-uniform Shishkin mesh in space. We prove that the numerical method is uniformly convergent of first order in time and almost second order in spatial variable. Numerical experiments are presented to validate the theoretical results.

1. Introduction

We consider a singularly perturbed linear system of parabolic initial-boundary value problems with a discontinuous source term on the domain $G = (0,1) \times (0,T]$. A single discontinuity in the source term is assumed to occur across the line $x = d$, $0 < d < 1$. Let $G_1 = (0,d) \times (0,T]$ and $G_2 = (d,1) \times (0,T]$. The corresponding initial-boundary value problem is to find $u \in C^0(\overline{G}) \cap C^{1+r}(G) \cap C^{4+r}(G_1 \cup G_2)$, $0 < r \le 1$:

$$\boldsymbol{L}\boldsymbol{u} := \frac{\partial \boldsymbol{u}}{\partial t} - \boldsymbol{E}\frac{\partial^2 \boldsymbol{u}}{\partial x^2} + \boldsymbol{A}\boldsymbol{u} = \boldsymbol{f}, \qquad (x,t) \in G_1 \cup G_2, \tag{1}$$

$$\boldsymbol{u}(0,t) = \boldsymbol{0}, \quad \boldsymbol{u}(1,t) = \boldsymbol{0}, \quad \forall\, t \in (0,T], \quad \boldsymbol{u}(x,0) = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{E} = \mathrm{diag}(\varepsilon_1,\dots,\varepsilon_m)$ with small parameters $\varepsilon_1,\dots,\varepsilon_m$ are such that $0 < \varepsilon_1 < \cdots < \varepsilon_m \ll 1$,

$$\boldsymbol{A}(x,t) = (a_{ij}(x,t))_{m\times m} \quad \text{and} \quad \boldsymbol{f}(x,t) = (f_i(x,t))_{m\times 1}. \tag{3}$$

$\boldsymbol{A}$ is smooth for all x and $\boldsymbol{f}$ is discontinuous at $x = d$, are given. We assume that the coupling matrix satisfies the following conditions:

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$$a_{ii}(x,t) > \sum_{j\neq i, j=1}^{m} |a_{ij}(x,t)|, \text{ for } 1\le i\le m, \text{ and } a_{ij}(x,t)\le 0 \text{ for } i\neq j, \tag{4}$$

and for some constant α, we have

$$0<\alpha<\min_{(x,t)\in\overline{G},1\le i\le m}\sum_{j=1}^{m}(a_{ij}(x,t)). \tag{5}$$

The solution $u=(u_1,\dots,\dots,u_m)^T$ satisfies the following interface condition $[u_i]=0$, $\varepsilon_i[\frac{\partial u_i}{\partial x}]=0$, at $x=d$.

For a good analytical discussion on singular perturbation problems and their numerical analysis, one may refer to [1], [4], [11]. In recent years, there has been renewed interest in coupled system of singularly perturbed problems. Shishkin [12] first studied a system of two parabolic partial differential equations on an infinite strip and proved the parameter-uniform rate of convergence was at least $O(N^{-1/4})$ using standard finite difference method on piecewise uniform meshes, when the two perturbation parameters were small and of different magnitude. Numerical methods for one dimensional time dependent singularly perturbed reaction–diffusion problems were given in [5–8]. Linss and Madden [8] discretized the problem using the central difference scheme in space and backward Euler in time and obtained first order parameter-uniform convergence for Bakhvalov meshes and almost first order uniform convergence for Shishkin meshes. Gracia et al. [6] considered an arbitrary number of equations and used a new decomposition of the solution which gave the asymptotic behavior of the solution to analyze the uniform convergence of the numerical approximations generated from the finite difference scheme, which was first order in time and essentially first order in space. Paramasivam et al. [5] considered the same problem and proved the first order in time and essentially second order in space. All these works considered continuous source term. Dunne and Riordan [2] considered several classes of singularly perturbed scalar parabolic differential equations in one space variable with discontinuous coefficients. Falco and Riordan [3] considered reaction–diffusion equation with a discontinuous diffusion coefficient. They also allowed the diffusion coefficient ε to be variable, and to be possibly discontinuous. Rao and Chawla [10] considered a coupled system of two singularly perturbed linear reaction–diffusion equations with discontinuous source term. The scheme was proved to be almost first order uniformly convergent in which the diffusion parameter associated with each equation of the system has a different order of magnitude. Paramasivam et al. [9] proved the result for an arbitrary number of equations. In this work, we present a parameter uniformly convergent numerical method for a singularly perturbed linear parabolic system with discontinuous source term, in which the diffusion term in each equation is multiplied by a small positive parameter of different magnitude.

The paper is organized as follows. Section 2 presents the continuous maximum principle, stability result, bounds on the exact solution and bounds on the regular and singular components. In Section 3 the piecewise-uniform variant of Shishkin mesh and the difference scheme that approximates the parabolic singularly perturbed problem is constructed. Error analysis is given in Section 4. Results of numerical experiments are presented in Section 5 for the validation of the theoretical results.

Notations We use C to denote a generic positive constant and $\boldsymbol{C}=(C,C,\dots,C)^T$ to denote a generic positive constant vector which are independent of the perturbation parameters and the discretization parameters N and M, but may not be same at each occurrence. Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two vector valued functions, we write $\boldsymbol{v}\le\boldsymbol{w}$ if $v_i\le w_i$, for $1\le i\le m$. For any functions $g, y_p\in C(\overline{\Omega})$, define $g_{i,j}=g(x_i,t_j)$, $y_{p;i,j}=y_p(x_i,t_j)$; if $\boldsymbol{g},\boldsymbol{y}_p\in C(\overline{\Omega})^m$ then $\boldsymbol{g}_{i,j}=\boldsymbol{g}(x_i,t_j)=(g_{1;i,j},\dots,g_{m;i,j})^T$, $\boldsymbol{y}_{p;i,j}=\boldsymbol{y}_p(x_i,t_j)=(y_{p;1;i,j},\dots,y_{p;m;i,j})^T$. We consider the maximum norm and denote it by $\|\,.\,\|_S$, where S is a closed and bounded set. We define $\|\,v\,\|_S=\max_{x\in S}|v(x)|$ and $\|\,\boldsymbol{v}\,\|_S=\max\{\|\,v_1\,\|_S,\|\,v_2\,\|_S,\dots,\|\,v_m\,\|_S\}$.

2. Properties of the exact solution

Theorem 2.1. *Let $\boldsymbol{A}(x,t)$ satisfy (4)–(5). Suppose that a function $u_i\in C(\overline{G})\cap C^2(G_1\cup G_2)$ satisfies $u_i(x,t)\ge 0$, $(x,t)\in\overline{G}\setminus G$, $[\boldsymbol{u}_x](d,t)\le\boldsymbol{0}$, $t>0$, $\frac{\partial u_i}{\partial t}\in C(G)$; $\boldsymbol{Lu}(x,t)\ge\boldsymbol{0}$, for all $(x,t)\in G_1\cup G_2$, then $\boldsymbol{u}(x,t)\ge\boldsymbol{0}$ for all $(x,t)\in\overline{G}$.*

Proof. Let $u_i(p_i)=\min_{(x,t)\in\overline{G}}\{u_i(x,t)\}$, for $1\le i\le m$. Assume without loss of generality $u_1(p_1)\le u_i(p_i)$, for $2\le i\le m$. If $u_1(p_1)\ge 0$, then there is nothing to prove. Suppose that $u_1(p_1)<0$, then the proof is completed by showing that this leads to a contradiction. Note that $p_1\notin\{\overline{G}\setminus G\}$. So either $p_1\in G_1\cup G_2$ or $p_1=(d,t_1)$.

In the first case,

$$\begin{aligned}(\boldsymbol{Lu})_1(p_1)&=\frac{\partial u_1}{\partial t}(p_1)-\varepsilon_1\frac{\partial^2 u_1}{\partial x^2}(p_1)+\sum_{j=1}^{m}a_{1j}(p_1)u_j(p_1)\\&=\frac{\partial u_1}{\partial t}(p_1)-\varepsilon_1\frac{\partial^2 u_1}{\partial x^2}(p_1)+\sum_{j=1}^{m}a_{1j}(p_1)u_j(p_1)+\sum_{j=1}^{m}a_{1j}(p_1)u_1(p_1)-\sum_{j=1}^{m}a_{1j}(p_1)u_1(p_1)<0.\end{aligned}$$