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Stability equivalence between the neutral delayed stochastic differential equations and the Euler–Maruyama numerical scheme ☆

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ABSTRACT

In this paper, the equivalence theorem for the mean square exponential stability between the neutral delayed stochastic differential equations (NDSDEs) and the Euler–Maruyama numerical scheme is investigated via the continuous time Euler–Maruyama solutions. Firstly, with some preliminaries on basic notations and assumptions, we establish the approximation degree of the numerical scheme to the underlying NDSDE under the global Lipschitz condition for the dynamics and contractive mapping condition for the neutral operator of the equation, which guarantee the existence and uniqueness of the global solution. Then we show that the underlying NDSDE is exponentially stable in mean square if and only if, for some sufficiently small stepsize, the Euler–Maruyama numerical scheme is exponentially stable in mean square. With such a theoretical result, the mean square exponential stability of NDSDEs can be affirmed just by the simulation approach in practice. Finally, a constructive example is proposed to verify the theoretical result by simulation. Relatively, some analysis around the present topic will be given by remarks and some challenging problems for further works will be proposed in the conclusion section.

1. Introduction

As is well known, numerical computations and simulations of differential equations and dynamical systems are important research issues for the engineering problems (e.g. those in control theory) and the ecological and economic ones. Actually the numerical computations and simulations provide auxiliary and intuitional approaches for analyzing the dynamical behaviors (e.g. the stability properties) of the underlying systems by digits and diagrams. To this end, the dynamical behaviors, e.g. the stability properties, of the underlying systems, the corresponding numerical schemes and the interrelations between them should be investigated. In this aspect, a key problem is: may we infer the dynamical properties of the underlying systems just by the numerical analysis or the simulation approach logically? Actually the numerical analysis or the simulation will make no sense if there is no such inferences. Accordingly, how do we guarantee this? Naturally there are many researches on these topics for various deterministic models in the past and many good results have been ob-

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tained in literature. While it has been found to be a difficult problem for the stochastic differential equations (SDEs) by the researchers.

Firstly, the stability properties of various numerical schemes for SDEs are investigated with contrast to those of the analytic models and a lot of results have been obtained in the literature, cf. e.g. [1,5–7,9,15,18,20,27,28]. Occasionally, this kind of results are achieved for both SDEs and the numerical schemes concurrently under the same assumptions, no logic relationship is established for the inclusion properties between the underlying equations and the numerical schemes. Accordingly one is unable to have a strict judgment on the stability properties for the underlying equations via the numerical computation and simulation approaches.

Secondly, some "if and only if" results for linear SDEs can be founded in some previous literature [4,29], those works shew that, the exponential stability in mean square of the numerical schemes of the underlying SDEs infers logically the corresponding exponential stability in mean square of the underlying SDEs, and vice versa, namely there is a mutual inferring with the stability properties between the linear SDEs and the numerical schemes. A classical work had thenceforth been made by [5]. Actually the techniques used in [5] have been known and may be regarded as quite standard now.

Thirdly, a general substantial progress had been made in [21] recently, which shew that the moment exponential stability of the stochastic differential equations and the numerical scheme can be mutually inferred from each other under appropriate conditions, which can be guaranteed by some basic and natural premise, for example, the global Lipschitz condition. Actually, in such a work, the approximation degree of the numerical scheme to the underlying equation plays a key role indeed, which implies the intrinsic and logic relationship between the numerical scheme and underlying equation. Thereafter, this work has been generalized to the delayed stochastic differential equations (DSDEs) in [24] etc. In the authors' opinion, the intrinsic interrelation between the underlying equations and the numerical schemes is obviously necessary for the purpose of mutual inference, which may come from either the construction and parameters of the underlying equations indirectly or the approximation degrees of the schemes directly, otherwise, one is unable to realize the mutual inference between the schemes and underlying equations due to that they are totally decoupled from each other.

Then how about the neutral delayed stochastic differential equations?

Actually, as a kind of more general, more interesting and more complex models than DSDEs, NDSDEs have received an extensive attention from researchers in the fields of applied mathematics, physics, biology, finance and engineering, and a lot of results have been obtained for the fundamental problem, the stability and control issues. Especially, as a basis for the present paper's topic, the stability of NDSDEs has been investigated widely in the past years and many good results have been obtained, see e.g. [2,8,10–14,17,18,22] etc. and the references therein. Of course, they are obviously more difficult to be dealt with than DSDEs due to the appearance of the neutral terms in the equations. At the same time, the stability issue also has been investigated widely for the numerical schemes of retarded and neutral stochastic differential equations with contrast to the underlying equations and a plenty of results have been obtained, cf. e.g. [16,22,23,25,26], and various kinds of stabilities have been considered in the literature, for example, the moment stability (M-stability), the almost sure stability, and the trajectory stability (T-stability) etc., see e.g. [1,7,20] etc. Especially, it should be pointed out that, the stability theorems and criteria are usually established by the Lyapunov function approach, and no logic relationship between the stability properties of the numerical schemes and underlying equations has been established really. Obviously this kind of logic mutual inferring can not be achieved by the Lyapunov function approach, due to that, in the authors' opinion, the Lyapunov function method provides only an assistant approach for us to determine stability properties of the underlying equations and the numerical schemes individually, while it is difficult to be applied to determine the interrelation between them. Thus we will follow the train of thought proposed by Mao etc. in [5,21] in the present paper.

Motivated by the above analysis, in this paper, we will study the exponential stability equivalence between NDSDEs and the Euler–Maruyama numerical scheme by the approximation degree method. Firstly, we establish the mean square approximation degree or say local convergence of the Euler–Maruyama numerical scheme under quite general conditions, that is, the global Lipschitz condition for the dynamics of the underlying equation and the contractive mapping condition for the neutral operator, which guarantee the existence and uniqueness of the global solution. Then we show that the underlying NDSDE is exponentially stable in mean square if and only if, for some sufficiently small step size, the Euler–Maruyama scheme is exponentially stable in mean square. By this conclusion, we can affirm mean square stability just by the simulation approach afterwards.

Interestingly our work will reveal that the investigation on the stability equivalence topic for the NDSDE is different from that for the retarded stochastic differential equations due to the appearance of the neutral terms, which bring about some additional condition and more difficulties for the derivations and arguments, and the initial data have their effects on the stability equivalence. Thus this paper is not simply a trivial generalization of the existing works to the more complex models.

The rest of the paper is organized as follows. Section 2 presents preliminaries for the whole work, including the basic notations, the equation description and the fundamental assumptions. In Section 3, we establish some fundamental lemmas which are useful in the next text and further works. In Section 4, we will establish the Euler–Maruyama numerical scheme, define the corresponding continuous approximate solution and establish several estimates around the continuous time approximate solution and the analytic solution, as a basis to carry out the main result. In Section 5, we will present our main result, namely we will show that the NDSDE is exponentially stable in mean square if and only if, for some sufficiently small stepsize, the Euler–Maruyama scheme is exponentially stable in mean square. In Section 6, we construct an