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Computing eigenvalues and eigenfunctions of the Laplacian for convex polygons

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ABSTRACT

Recently a new transform method, called the Unified Transform or the Fokas method, for solving boundary value problems (BVPs) for linear and integrable nonlinear partial differential equations (PDEs) has received a lot of attention. For linear elliptic PDEs, this method yields two equations, known as the global relations, coupling the Dirichlet and Neumann boundary values. These equations can be used in a collocation method to determine the Dirichlet to Neumann map. This involves expanding the unknown functions in terms of a suitable basis, and choosing a set of collocation points at which to evaluate the global relations. Here, using these methods for the Helmholtz and modified Helmholtz equations and following the earlier results of [15], we determine eigenvalues of the Laplacian in a convex polygon. Eigenvalues are characterised by the points where the generalised Dirichlet to Neumann map becomes singular. We find that the method yields spectral convergence for eigenfunctions smooth on the boundary and for non-smooth boundary values, the rate of convergence is determined by the rate of convergence of expansions in the chosen Legendre basis. Extensions to the case of oblique derivative boundary conditions and constant coefficient elliptic PDEs are also discussed and demonstrated.

1. Introduction

In the late nineties, a new method for analysing boundary-value problems (BVP) for linear and for integrable nonlinear partial differential equations (PDEs) was introduced by the second author [16–18]. This method, which has become known as the unified transform or the Fokas method, has been applied to a variety of linear elliptic PDEs formulated in the interior (and exterior) of a polygon. One can think of the method as a generalised Fourier transform, each transform being tailored to the PDE at hand. For the Laplace, modified Helmholtz and Helmholtz equations, the method expresses the solution in terms of integrals in the complex Fourier plane. However, these integrals contain integral transforms of both the Dirichlet and Neumann boundary values. These representations are analogous to the classical Green's representations, but they are formulated in the Fourier space as opposed to physical space. The transforms of the Dirichlet and Neumann boundary values are coupled via two algebraic equations – the global relations. This method has been used to obtain solutions where classical methods fail [21,43] and has been put on a rigorous footing by Ashton [1,2].

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There has been considerable interest in using the global relations to evaluate numerically the generalised Dirichlet to Neumann map [20,22,15,23,37–40,42,27]. The general approach can be summed up in two steps. First, one expands the unknown boundary values in some suitable basis. The choice of the basis can significantly improve the convergence properties of the method. We will chose Legendre polynomials since their Fourier transform can be expressed via modified Bessel functions and they have been demonstrated to have good convergence properties for the method. Next, one evaluates the approximate global relation using these expansions at suitable collocation points. This gives a finite linear system of equations which, assuming the generalised Dirichlet to Neumann map is not singular, can be inverted for an approximation of the unknown boundary values. It is found that over-determining the system yields smaller condition numbers. Here, we take advantage of recent developments in this area [42,27] in our choice of collocation points. The method can also be adopted to form a Galerkin scheme by integrating the global relations [2,6].

In this paper, we use the above method to study the generalised Dirichlet to Neumann map when it becomes singular. It is shown that the points where the map becomes singular correspond precisely to eigenvalues. A scheme for computing these eigenvalues and eigenfunctions is introduced and its effectiveness is demonstrated on a range of examples with different boundary conditions. We find that the method yields spectral convergence, analogous to the convergence demonstrated for this method when inverting the Dirichlet to Neumann map. The eigenvalue/eigenfunction problem has many applications in engineering and physics [14,8,34] and in data analysis [36]. In two dimensions it is known as the 'drum' problem [29,24, 31,44]. There are only a handful of domains for which the solution is known analytically and hence a numerical approach is needed. Indeed, there is a vast literature on numerical methods for this problem [10,31]. In general, obtaining eigenvalues of elliptic PDEs with standard methods, such as finite element or finite difference, is problematic, yielding only algebraic convergence and becoming impractical for large eigenvalues. The method presented here yields a diagonally dominant matrix, which is much smaller than standard discretisation methods, and its size in general grows linearly with the number of basis functions used. Furthermore, the method avoids completely the issue of evaluating singular integrals, which appear in boundary-integral equation methods and corresponding boundary-based discretisation methods. An apparent deficiency of the new method is the need to analyse a non linear eigenvalue problem. In practice, this is not an issue since the method is very easily parallelisable. We also find that very small system sizes are needed to obtain high accuracy in spectral data. The idea of using the Fokas method to compute eigenvalues was first presented in [15] in the case of a trapezium. Here, we implement this idea to a variety of examples, compute eigenfunctions and demonstrate that the convergence rate is determined by the convergence rate of the Legendre basis expansion.

The paper is organised as follows: in Section 2 we introduce the global relation and show that non trivial solutions correspond precisely to eigenvalues and eigenfunctions. Section 3 introduces the discretisation method, the choice of collocation points and reviews other methods found in the literature. In Section 4, we demonstrate the method on examples with known spectral data, compare to the finite element method and also demonstrate how eigenvalue multiplicities and eigenspaces can be computed. Section 5 analyses examples with unknown spectrum where we demonstrate convergence rates expected from the asymptotics of corner singularities, whereas Section 6 uses the modified Helmholtz equation to deal with negative eigenvalues. In Section 7, we extend the method to more general constant coefficient elliptic PDEs, and Section 8 discusses further the results presented in this paper.

2. Global relation

Our aim is to solve the eigenvalue problem

$$-u_{xx} - u_{yy} = \lambda u, \quad (x, y) \in \Omega, \tag{1}$$

$$\delta_j u_j^{\mathcal{N}} - \gamma_j u_j = 0 \quad \delta_j = \sqrt{1-\gamma_j^2}, \quad \gamma_j \in [-1, 1], \quad j = 1, ..., n, \tag{2}$$

for some bounded convex polygon Ω with sides $\{\Gamma_j\}_{j=1}^n$, where $u_j^{\mathcal{N}}$ denotes the normal derivative along side j. The constants $\{\gamma_j\}$ prescribe Robin boundary conditions along each side, which can be made precise in a trace sense. It is known, that these boundary conditions give rise to a self-adjoint operator, bounded below with compact resolvent whose eigenfunctions from a complete basis of $L^2(\Omega)$ (see [33] Theorem 4.12). The sign of the eigenvalues depends on the sign of γ_j. Dirichlet, Neumann or negative γ_j give rise to non-negative eigenvalues (formally justified by integration by parts). We shall be predominantly interested in this case, but we will demonstrate that our method also works for negative eigenvalues by using the modified Helmholtz equation. For non-negative eigenvalues we may write $\lambda = k^2$ and recast the eigenvalue problem as a non-zero solution to the Helmholtz equation

$$u_{xx} + u_{yy} + k^2 u = 0, \quad (x, y) \in \Omega, \tag{3}$$

$$\delta_j u_j^{\mathcal{N}} - \gamma_j u_j = 0, \quad j = 1, ..., n. \tag{4}$$

We briefly recall the unified transform method used to solve the Helmholtz equation [20,5]. Letting $z = x + iy$ and $\overline{z} = x - iy$ we have that $V = \exp\left((-ik/2)[\lambda z + \overline{z}/\lambda]\right)$ is a solution of the Helmholtz equation for all $\lambda \in \mathbb{C}\backslash\{0\}$. This allows us to write the PDE in divergence form,