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Increasing the approximation order of the triangular Shepard method

F. Dell'Accio [a], F. Di Tommaso [a,*], O. Nouisser [b], B. Zerroudi [b]

[a] *Department of Mathematics and Computer Science, University of Calabria, Rende, Italy*
[b] *Department of Mathematics, Université Ibn Tofail, Kenitra, Morocco*

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ABSTRACT

In this paper we discuss an improvement of the triangular Shepard operator proposed by Little to extend the Shepard method. In particular, we use triangle based basis functions in combination with a modified version of the linear local interpolant on the vertices of the triangle. We deeply study the resulting operator, which uses functional and derivative data, has cubic approximation order and a good accuracy of approximation. Suggestions on how to avoid the use of derivative data, without losing both order and accuracy of approximation, are given.

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1. Shepard and triangular Shepard operators

In 1983 Little introduces a variation of the Shepard method, one of the earliest technique for interpolating scattered data $\{(x_i, f_i)\}_{i=1,\dots,n}$, where $X = \{x_1, x_2, \dots, x_n\} \subset \mathbb{R}^2$ has no-structure and $f_i = f(x_i)$ are the values of an unknown function f. Little associates to the node set X an appropriate list of triangles T, with vertices $\{x_{j_1}, x_{j_2}, x_{j_3}\} \subset X$ such that $X = \bigcup_{j_1, j_2, j_3} \{x_{j_1}, x_{j_2}, x_{j_3}\}$, and substitutes the point-based Shepard basis functions

$$A_{\mu,i}(x) = \frac{\dfrac{1}{|x - x_i|^{\mu}}}{\displaystyle\sum_{k=1}^{n} \frac{1}{|x - x_k|^{\mu}}}, \qquad i = 1, \dots, n \tag{1}$$

with triangle-based basis functions

$$B_{\mu,j}(x) = \frac{\displaystyle\prod_{\ell=1}^{3} \frac{1}{|x - x_{j_\ell}|^{\mu}}}{\displaystyle\sum_{k=1}^{m} \prod_{\ell=1}^{3} \frac{1}{|x - x_{k_\ell}|^{\mu}}}, \qquad j = 1, \dots, m, \tag{2}$$

* Corresponding author.
E-mail addresses: francesco.dellaccio@unical.it (F. Dell'Accio), ditommaso@mat.unical.it (F. Di Tommaso), otheman.nouisser@yahoo.fr (O. Nouisser), zerroudi@gmail.com (B. Zerroudi).

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where $|\cdot|$ is the euclidean norm in $\mathbb{R}^2$ and μ is a positive parameter. In addition, Little substitutes the functional values f_i in the convex combination defining the Shepard operator

$$S_\mu[f](x) = \sum_{i=1}^{n} A_{\mu,i}(x) f_i \tag{3}$$

with the values of the local linear interpolant $L_j[f](x)$ on the vertices $\{x_{j_1}, x_{j_2}, x_{j_3}\}$ of the triangle $t_j \in T$ and introduces the triangular Shepard operator

$$K_\mu[f](x) = \sum_{j=1}^{m} B_{\mu,j}(x) L_j[f](x). \tag{4}$$

As remarked by Little, the advantages of the triangular Shepard operator (4) with respect to the Shepard operator (3) are a higher polynomial precision and a better esthetic behavior in approximating and interpolating continuous functions when only functional data values are supplied. In fact both Shepard basis function (1) and triangular Shepard basis functions (2) are positive

$$A_{\mu,i}(x) \geq 0, i = 1, \ldots, n, \qquad B_{\mu,j}(x) \geq 0, j = 1, \ldots, m, \tag{5}$$

and form a partition of unity

$$\sum_{i=1}^{n} A_{\mu,i}(x) = 1, \qquad \sum_{j=1}^{m} B_{\mu,j}(x) = 1. \tag{6}$$

Moreover they are cardinal, that is

$$A_{\mu,i}(x_j) = \begin{cases} 0, & \textit{if } i \neq j, \\ 1, & \textit{if } i = j, \end{cases} \tag{7}$$

while

$$B_{\mu,j}(x_i) = 0, \text{ for each } x_i \notin \{x_{j_1}, x_{j_2}, x_{j_3}\}, \tag{8}$$

and

$$\sum_{j \in J_i} B_{\mu,j}(x_i) = 1, \tag{9}$$

where $J_i = \left\{k \in \{1, \ldots, m\} : i \in \{k_1, k_2, k_3\}\right\}$ is the set of indices of all triangles which have x_i as a vertex. But, while the vanishing of the first order derivatives of the Shepard basis functions ($\mu > 1$) at each interpolation node

$$\nabla A_{\mu,i}(x_j) = 0, \quad \text{for each } x_j, \tag{10}$$

causes the presence of flat spots in the Shepard interpolant, the analogous property satisfied by the triangular Shepard basis functions ($\mu > 1$)

$$\nabla B_{\mu,j}(x_i) = 0, \quad \text{for each } x_i \notin \{x_{j_1}, x_{j_2}, x_{j_3}\}, \tag{11}$$

and

$$\sum_{j \in J_i} \nabla B_{\mu,j}(x_i) = 0, \tag{12}$$

allows, in the triangular Shepard operator, the interpolation of the first order derivatives of the *local linear interpolant* at each node [5].

On the other hand, at a first glance, Little remarks that the triangular Shepard is certainly more complex than Shepard's method since it presupposes the definition of an appropriate list of index triples and he does not give any suggestion on how to choose triangles, neither studies the approximation order nor the accuracy of approximation of the introduced operator. A recent paper [5] highlights the good approximation accuracy of the triangular Shepard method and its quadratic approximation order ($\mu > 4/3$) for both Delaunay triangulations of the nodes and compact triangulations with about 65% fewer triangles which yields a more efficient interpolation operator with comparable approximation accuracy. In fact, similarly to the Delaunay triangulation, compact triangulations avoid slider triangles, since they are realized in order to minimize the error of the local interpolants. But unlike the Delaunay triangulations, the procedure to define a compact triangulation is local, that is all triangles are independent each from others, to the point that overlapping or disjoint triangles are allowed.