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A pseudospectral scheme and its convergence analysis for high-order integro-differential equations

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ABSTRACT

The main purpose of this work is to develop an integral pseudospectral scheme for solving integro-differential equations. We provide new pseudospectral integration matrices (PIMs) for the Legendre–Gauss and the flipped Legendre–Gauss–Radau points, respectively, and present an efficient and stable approach to computing the PIMs via the recursive calculation of Legendre integration matrices. Furthermore, we provide a rigorous convergence analysis for the proposed pseudospectral scheme in both L^∞ and L^2 spaces via a linear integral equation, and the spectral rate of convergence is demonstrated by numerical results.

1. Introduction

A variety of phenomena can be modeled as integro-differential equations (IDEs). By far a large collection of numerical methods has been developed to solve those equations (see, e.g., [3,5,7,10,13,14,19,20]), among which pseudospectral methods have been one of the most promising methods. The main advantage of pseudospectral methods over other methods lies in their ability to provide highly accurate approximations while converging spectrally (i.e., at an exponential rate) for smooth problems, exhibiting the so called "spectral accuracy" [16]. In general, pseudospectral methods can be grouped into two major categories: differential and integral. The former are known to be severely ill-conditioned when the number of collocation points is large, while the latter are well-conditioned even for large number of collocation points [8,10]. In this work, we focus on integral pseudospectral methods.

The basic principle of integral pseudospectral methods is to recast IDEs as integral equations (IEs), and then discretize the latter using pseudospectral integration matrices (PIMs) [9]. Therefore, the well-conditioning of integral pseudospectral methods relies heavily on that of PIMs and several distinctive approaches [6,8,9] have been proposed for computing PIMs. El-Gendi et al. [6] adopted the Cauchy's formula to compute the PIM for the Chebyshev–Gauss–Lobatto (CGL) points from the corresponding first-order PIM. Subsequently, Elbarbary [8] presented a new approach to computing the same PIM based on the exact relation between the Chebyshev polynomials and their derivatives. Recently, Elgindy [9] has derived the above PIM using an explicit formula for the iterated integrals of Chebyshev polynomials.

The aim of this paper is to develop an integral pseudospectral scheme for solving IDEs. More precisely, we provide new PIMs for the Legendre–Gauss (LG) and the flipped Legendre–Gauss–Radau (FLGR) points, respectively, and present an efficient and stable approach to computing the PIMs via the recursive calculation of Legendre integration matrices (LIMs). Furthermore, we provide a rigorous convergence analysis for the proposed pseudospectral scheme in both L^∞ and L^2 spaces via a linear IE, which theoretically justifies the spectral rate of convergence.

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The rest of this paper is organized as follows. In Section 2, the Legendre polynomials are presented for subsequent developments. In Section 3, the definitions and computation of LIMs are presented. This is followed by the definitions and computation of PIMs in Section 4. The detailed implementation of integral pseudospectral scheme for solving IDEs is provided in Section 5. In Section 6, the convergence analysis in both L^∞ and L^2 spaces is provided. Numerical results are shown in Section 7. Finally, Section 8 contains some concluding remarks.

2. Legendre polynomials

The Legendre polynomials are orthogonal polynomials on the interval $[-1, +1]$, and satisfy the following orthogonality relation:

$$\int_{-1}^{+1} L_i(\tau) L_j(\tau)\,\mathrm{d}\tau = \lambda_i \delta_{ij}, \tag{1}$$

where $\lambda_i = \frac{2}{2i+1}$ is the normalization factor, and δ_{ij} is the Kronecker delta. The three-term recursion formula for the Legendre polynomials is given by

$$L_0(\tau) = 1, \quad L_1(\tau) = \tau, \tag{2a}$$

$$L_{n+1}(\tau) = \frac{(2n+1)\tau L_n(\tau) - nL_{n-1}(\tau)}{n+1}, \quad n = 1, 2, \ldots, \tag{2b}$$

which yields an efficient and stable way to evaluate the Legendre polynomials of arbitrary order at any $\tau \in [-1, +1]$.

3. Definitions and computation of LIMs

In this section, the definitions and computation of LIMs for the LG and FLGR points are presented, respectively.

3.1. Definitions of LIMs

Definition 1. The LIM of order $\ell \geq 1$ for the LG points of $\{\tau_i \in (-1, +1)\}_{i=1}^N$ with $-1 = \tau_0 < \tau_1 < \cdots < \tau_{N+1} = +1$ is defined as

$$\boldsymbol{L}_{kj}^{\ell} \triangleq \underbrace{\int_{-1}^{\tau_k}\int_{-1}^{\sigma_\ell} \cdots \int_{-1}^{\sigma_3}\int_{-1}^{\sigma_2}}_{\ell} L_j(\sigma_1)\,\mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \cdots \mathrm{d}\sigma_{\ell-1}\mathrm{d}\sigma_\ell, \quad (k = 1, 2, \ldots, N+1, \quad j = 0, 1, \ldots, N-1). \tag{3}$$

Definition 2. The LIM of order $\ell \geq 1$ for the FLGR points of $\{\hat{\tau}_i \in (-1, +1]\}_{i=1}^N$ with $-1 = \hat{\tau}_0 < \hat{\tau}_1 < \cdots < \hat{\tau}_N = +1$ is defined as

$$\hat{\boldsymbol{L}}_{kj}^{\ell} \triangleq \underbrace{\int_{-1}^{\hat{\tau}_k}\int_{-1}^{\sigma_\ell} \cdots \int_{-1}^{\sigma_3}\int_{-1}^{\sigma_2}}_{\ell} L_j(\sigma_1)\,\mathrm{d}\sigma_1 \mathrm{d}\sigma_2 \cdots \mathrm{d}\sigma_{\ell-1}\mathrm{d}\sigma_\ell, \quad (k = 1, 2, \ldots, N, \quad j = 0, 1, \ldots, N-1). \tag{4}$$

3.2. Computation of LIMs

Now, we present an efficient and stable approach to computing the LIMs defined above. The main results are given in the following theorem.

Theorem 1. *The LIM of order $\ell \geq 2$ for the LG points $\{\tau_i\}_{i=1}^N$ can be computed exactly as*

$$\boldsymbol{L}_{kj}^{\ell} = \begin{cases} \boldsymbol{L}_{k1}^{\ell-1} + \boldsymbol{L}_{k0}^{\ell-1}, & j = 0 \\ \frac{1}{2j+1}\left(\boldsymbol{L}_{k,j+1}^{\ell-1} - \boldsymbol{L}_{k,j-1}^{\ell-1}\right), & j \geq 1 \end{cases}, \tag{5}$$

where