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The second order perturbation approach for elliptic partial differential equations on random domains ☆

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ABSTRACT

The present article is dedicated to the solution of elliptic boundary value problems on random domains. We apply a high-precision second order shape Taylor expansion to quantify the impact of the random perturbation on the solution. Thus, we obtain a representation of the solution with third order accuracy in the size of the perturbation's amplitude. The major advantage of this approach is that we end up with purely deterministic equations for the solution's moments. In particular, we derive representations for the first four moments, i.e., expectation, variance, skewness and kurtosis. These moments are efficiently computable by means of boundary integral equations. Numerical results are presented to validate the presented approach.

1. Introduction

Often, practical problems from science and engineering result in the task of solving a boundary value problem for an unknown function. The numerical solution of such boundary value problems is in general well understood, at least if the problem's input parameters are known exactly. Often, however, the input parameters are not known exactly. Hence, the challenge is to obtain high-precision approximations also in the presence of uncertainties. Typically, random input parameters are then modeled in terms of random fields and, as a consequence, the given boundary value problem is turned into a random one. This yields a solution which is a random field itself. In this article, to keep the presentation simple for the reader's convenience, we shall consider the Dirichlet problem for the Poisson equation which is formulated relative to a random domain:

$$-\Delta u(\omega) = f \text{ in } D(\omega), \quad u(\omega) = g \text{ on } \partial D(\omega). \tag{1}$$

Here, $D(\omega)$ denotes the domain under consideration with boundary $\partial D(\omega)$ which both depend on the random parameter $\omega \in \Omega$. Of course, this problem can be easily extended for the case of more complex data, as for example a more complex diffusion coefficient, cf. [16]. The idea of taking random computational domains into account is inspired by tolerances in the fabrication process of a mechanical device or by damages of the boundary which appear during the life cycle of a device. Typically, such devices are close to a nominal shape but differ of course from its mathematical definition. Since these

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tolerances are in general small, we can also make the crucial assumption of the smallness of the random perturbations. Uncertainty quantification for computational domains arouses recently more interest, see [4,13,14,17,24,29].

By identifying domains with their boundary, a random domain $D(\omega)$, which is close to a given nominal domain D_0, can be described as a normal perturbation of this nominal boundary ∂D_0:

$$\partial D(\omega) = \{\mathbf{y} \in \mathbb{R}^n : \mathbf{y}(\mathbf{x}) = \mathbf{x} + \varphi(\mathbf{x}, \omega)\mathbf{n}(\mathbf{x}),\ \mathbf{x} \in \partial D_0\}. \tag{2}$$

In this context, the random field $\varphi(\omega)\colon \partial D_0 \to \mathbb{R}$ is a scalar function which is defined with respect to the nominal boundary ∂D_0. It uniquely determines the domain perturbation via $\varphi(\omega)\mathbf{n}\colon \partial D_0 \to \mathbb{R}^n$, with $\mathbf{n}$ denoting the outward normal to the domain D_0.

The most simple methodology to deal with randomness in numerical computations is the Monte-Carlo method, cf. [22,27]. Here, numerous draws of the random input data are sampled according to some a-priori known or empirical distribution. Each draw entails the computation of a deterministic boundary value problem. Then, the statistics like the mean and the variance of these samples are formed. Nevertheless, for boundary value problems on random domains, each sample implies a new domain and thus a new mesh, the assembly of new mass and stiffness matrices, etc. Therefore, the Monte-Carlo method is extremely costly and rather difficult to implement for the problem at hand. Note that the same accounts for other more sophisticated quadrature techniques like the quasi-Monte Carlo quadrature, cf. [3], or sparse quadrature methods, cf. [2].

Thus, we aim here at a different approach, namely the perturbation approach, see [1,13,14,17,20,21]. It facilitates to approximate the random solution on an arbitrary compactum inside the fixed nominal domain D_0. The pivotal idea of the perturbation approach for random boundary value problems is the expansion of the underlying random field around the related input parameter's expectation, in our case the domain D_0, via a (shape-) Taylor expansion. For the boundary value problem (1) at hand, this will involve shape calculus, cf. [7,25,28]. With the help of the shape Taylor expansion, we can derive asymptotic expansions of the random output's expectation, variance and also higher order moments.

More precisely, we employ a second order shape Taylor expansion and derive corresponding asymptotic expansions for the first four moments. These can be computed explicitly under the finite noise assumption. This means, the random domain perturbation in (2) is of the form

$$\varphi(\mathbf{x}, \omega) = \sum_{i=1}^{N} \varphi_i(\mathbf{x}) Y_i(\omega) \tag{3}$$

with centered random variables $Y_i\colon \Omega \to [-1, 1]$ which are independent and identically distributed, see [29,5].

As we will show, in the setting (3), i.e., having N terms in the series expansion of the random perturbation field $\varphi\mathbf{n}$, the expectation and the variance can be computed with a computational cost of order $\mathcal{O}(N)$. The skewness and kurtosis can be computed with a computational cost of order $\mathcal{O}(N^2)$.

We remark that a similar approach for scalar output functionals of partial differential equations on uncertain domains has already been considered in [6]. Such shape functionals can be linearized by means of shape calculus, which, in particular, involves the computation of the shape Hessian of the functional under consideration. However, employing the adjoint method, which is well-known in shape optimization, only the first order shape derivative of the partial differential equation under consideration has to be computed. Whereas, for the problem considered in the present article, also the second order shape derivative of the partial differential equation has to be computed. The latter is computationally much more demanding.

The rest of this article is organized as follows. In Section 2, we introduce the basic ideas of shape calculus and derive the asymptotic expansions for the random solution's statistics. Then, in Section 3, we propose a way to compute these expansions by means of a boundary element method. Numerical results are presented in Section 4. Finally, we state concluding remarks in Section 5.

2. Perturbation analysis

To avoid the extreme high-dimensionality of a direct discretization of (1) by means of the domain mapping method, see e.g. [29], which is driven by the size N of the expansion (3), a technique can be applied which is mainly known from shape sensitivity analysis, namely the so-called local shape derivative, see [8,26]. It has been established as a measure of the solution's dependence on domain or boundary perturbations. Such shape derivatives are in principle known since Hadamard, cf. [12] and nowadays well established in shape optimization, see [7,18,25,28]. Since the solution's nonlinear dependence on the shape of the domain is Fréchet differentiable, we can linearize it around the nominal domain D_0. Thus, deterministic expressions for the solution's statistics can be derived.

2.1. Shape calculus

Consider a sufficiently smooth domain D_0 and a boundary variation in the direction of the outward normal $\mathbf{n}$:

$$\varphi\mathbf{n}\colon \partial D_0 \to \mathbb{R}^n \quad \text{such that} \quad \|\varphi\|_{C^{2,1}(\partial D_0)} \le 1.$$