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A discrete divergence free weak Galerkin finite element method for the Stokes equations

Lin Mu [a,*,1], Junping Wang [b,2], Xiu Ye [c,3], Shangyou Zhang [d]

[a] Computer Science and Mathematics Division, Oak Ridge National Laboratory, Oak Ridge, TN, 37831, USA
[b] Division of Mathematical Sciences, National Science Foundation, Arlington, VA 22230, USA
[c] Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA
[d] Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA

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ABSTRACT

A discrete divergence free weak Galerkin finite element method is developed for the Stokes equations based on a weak Galerkin (WG) method introduced in [17]. Discrete divergence free bases are constructed explicitly for the lowest order weak Galerkin elements in two and three dimensional spaces. These basis functions can be derived on general meshes of arbitrary shape of polygons and polyhedrons. With the divergence free basis derived, the discrete divergence free WG scheme can eliminate pressure variable from the system and reduces a saddle point problem to a symmetric and positive definite system with many fewer unknowns. Numerical results are presented to demonstrate the robustness and accuracy of this discrete divergence free WG method.

1. Introduction

The Stokes problem seeks unknown functions $\mathbf{u}$ and p satisfying

$$-\nabla\cdot A\nabla\mathbf{u}+\nabla p=\mathbf{f}\quad \text{in }\Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=0\quad \text{in }\Omega, \tag{1.2}$$

$$\mathbf{u}=\mathbf{g}\quad \text{on }\partial\Omega, \tag{1.3}$$

where Ω is a polygonal domain in $\mathbb{R}^d$ with $d=2,3$ and A is a symmetric and positive definite $d\times d$ matrix-valued function in Ω.

The weak form in the primary velocity-pressure formulation for the Stokes problem (1.1)–(1.3) seeks $\mathbf{u}\in[H^1(\Omega)]^d$ and $p\in L_0^2(\Omega)$ satisfying $\mathbf{u}=\mathbf{g}$ on $\partial\Omega$ and

* Corresponding author.
E-mail addresses: mul1@ornl.gov (L. Mu), jwang@nsf.gov (J. Wang), xxye@ualr.edu (X. Ye), szhang@udel.edu (S. Zhang).
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$$(A\nabla\mathbf{u},\nabla\mathbf{v})-(\nabla\cdot\mathbf{v},p)=(\mathbf{f},\mathbf{v}),\quad \forall\mathbf{v}\in[H_0^1(\Omega)]^d, \tag{1.4}$$

$$(\nabla\cdot\mathbf{u},q)=0,\qquad \forall q\in L_0^2(\Omega). \tag{1.5}$$

In the standard finite element methods for the Stokes and the Navier–Stokes equations, both pressure and velocity are approximated simultaneously. The primitive system is a large saddle point problem. Numerical solvers for such indefinite systems are usually less effective and robust than solvers for definite systems. On the other hand, the divergence-free finite element method, discrete or exact, computes numerical solution of velocity by solving a symmetric positive definite system in a divergence-free subspace. It eliminates the pressure from the coupled equations and hence significantly reduces the size of the system. The divergence-free method is particularly attractive in the cases where the velocity is the primary variable of interest, for example, the groundwater flow calculation. The main tasks in the implementation of the divergence-free method are to understand divergence-free subspaces, weakly or exactly, and to construct bases for them.

Many finite element methods, continuous [3,4,9] and discontinuous [2,5,13,14,18], have been developed and analyzed for the Stokes and the Navier–Stokes equations. Divergence-free basis for different finite element methods have been constructed [1,6–8,10–12,19–21].

A weak Galerkin finite element method was introduced in [17] for the Stokes equations in the primal velocity-pressure formulation. This method is designed by using discontinuous piecewise polynomials on finite element partitions with arbitrary shape of polygons/polyhedra. Weak Galerkin methods were first introduced in [15,16] for second order elliptic equations. In general, weak Galerkin finite element formulations for partial differential equations can be derived naturally by replacing usual derivatives by weakly-defined derivatives in the corresponding variational forms, with an option of adding a stabilization term to enforce a weak continuity of the approximating functions. Therefore the weak Galerkin method developed in [17] for the Stokes equations naturally has the form: find $\mathbf{u}_h\in V_h$ and $p_h\in W_h$ satisfying $\mathbf{u}_h=Q_b\mathbf{g}$ on $\partial\Omega$ and

$$(A\nabla_w\mathbf{u}_h,\nabla_w\mathbf{v})+s(\mathbf{u}_h,\mathbf{v})-(\nabla_w\cdot\mathbf{v},p_h)=(\mathbf{f},\mathbf{v}), \tag{1.6}$$

$$(\nabla_w\cdot\mathbf{u}_h,q)=0 \tag{1.7}$$

for all the test functions $\mathbf{v}\in V_h^0$ and $q\in W_h$ where V_h and W_h will be defined later. The stabilizer $s(\mathbf{u}_h,\mathbf{v})$ in (1.6) is parameter independent.

Let D_h be a discrete divergence free subspace of V_h^0 such that $(\nabla_w\cdot\mathbf{v},q)=0$ for any $q\in W_h$. Then the discrete divergence free WG formulation is to find $\mathbf{u}_h$ satisfying $\mathbf{u}_h=Q_b\mathbf{g}$ on $\partial\Omega$ and

$$(A\nabla_w\mathbf{u}_h,\nabla_w\mathbf{v})+s(\mathbf{u}_h,\mathbf{v})=(\mathbf{f},\mathbf{v}),\quad\forall\mathbf{v}\in D_h. \tag{1.8}$$

System (1.8) is symmetric and positive definite with many fewer unknowns. The main purpose of this paper is to construct bases for D_h in two and three dimensional spaces. A unique feature of these divergence free basis functions is that they can be obtained on general meshes such as hybrid meshes or meshes with hanging nodes. Numerical examples in two dimensional space are provided to confirm the theory. Although the Stokes equations is considered, the divergence free basis can be used for solving the Navier–Stokes equations.

2. A weak Galerkin finite element method

In this section, we will review the WG method for the Stokes equations introduced in [17] with $k=1$.

Let $\mathcal{T}_h$ be a partition of the domain Ω consisting of a set of polyhedra satisfying a set of conditions specified in [16]. In addition, we assume that all the elements $T\in\mathcal{T}_h$ are convex. Denote by $\mathcal{F}_h$ the set of all edges in 2D or faces in 3D in $\mathcal{T}_h$, and let $\mathcal{F}_h^0=\mathcal{F}_h\backslash\partial\Omega$ be the set of all interior edges or faces.

The weak Galerkin methods introduce a new way to define a function v, called weak function, that allows v taking different forms in the interior and on the boundary of the element:

$$v=\begin{cases} v_0, & \text{in } T^0,\\ v_b, & \text{on } \partial T,\end{cases}$$

where T^0 is the interior of T. Since a weak function v is formed by two parts v_0 and v_b, we write v as $v=\{v_0,v_b\}$ in short without confusion.

We define a finite element space consisting of these weak functions for the velocity as follows

$$V_h=\left\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}:\ \mathbf{v}_0|_T\in[P_1(T)]^d,\ \mathbf{v}_b|_e\in[P_0(e)]^d,\ e\subset\partial T,\forall T\in\mathcal{T}_h\right\}.$$

Please note that $\mathbf{v}_b$ takes single value on $e\in\mathcal{F}_h$.

We define two subspaces of V_h,

$$V_h^0=\{\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}\in V_h:\ \mathbf{v}_b=0\text{ on }\partial\Omega\}.$$