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Differential equations for families of semi-classical orthogonal polynomials within class one

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ABSTRACT

In this paper we study families of semi-classical orthogonal polynomials within class one. We derive general second or third order ordinary differential equations (with respect to certain parameters) for the recurrence coefficients of the three-term recurrence relation of these polynomials and show that in particular well-known cases, e.g. related to the modified Airy and Laguerre weights, these equations can be reduced to the second and the fourth Painlevé equations.

1. Motivation

The subject matter of the present paper lies within two well-known topics of special functions: semi-classical orthogonal polynomials and the Painlevé equations.

Semi-classical orthogonal polynomials have been extensively studied in the literature, from many points of view [2,15,17]. A key feature of semi-classical orthogonal polynomials is the Pearson equation for the weight, $w'(z)/w(z) = R(z)$, where R is a rational function of z (see Section 2 for more details). A frequently encountered study in problems in Mathematical Physics is the analysis of modifications of semi-classical weights as functions of some parameters and their consequences for basic structures of the polynomials – the recurrence relation and the deformation derivatives. In this topic, the connections with the Painlevé equations are very well-known, showing that the three-term recurrence relation coefficients are often governed by equations of the Painlevé type (see, for instance, [1,5–7,9,14,15,21]).

Recall that the Painlevé equations (P_I)–(P_{VI}) are nonlinear second order ordinary differential equations having the property that all movable singularities of an arbitrary solution are at most poles (the so-called Painlevé property). See [8] for various properties, symmetries and application of the Painlevé equations. In this paper we shall deal with the second and the fourth Painlevé equations which are respectively given by

$$y'' = 2y^3 + zy + \alpha \qquad P_{II}(\alpha)$$

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and

$$y'' = \frac{y'^2}{2y} + \frac{3y^3}{2} + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}, \qquad P_{IV}(\alpha,\beta)$$

where α and β are arbitrary parameters. Note that by a simple scaling $y(z) \to iy(-iz)$, where $i^2=-1$, equation $P_{IV}(\alpha,\beta)$ is transformed to $P_{IV}(-\alpha,\beta)$. A similar scaling transformation exists also for the second Painlevé equation changing α to $-\alpha$. The second Painlevé equation has classical solutions expressible in terms of the Airy functions if and only if

$$\alpha = n+1/2, \quad n\in\mathbb{Z}.$$

The fourth Painlevé equation has special solutions expressible in terms of parabolic cylinder functions if and only if either

$$\beta = -2(2n+1+\varepsilon\alpha)^2, \quad \varepsilon^2=1, \; n\in\mathbb{Z}, \tag{1}$$

or

$$\beta = -2n^2, \quad n\in\mathbb{Z}. \tag{2}$$

We shall also need another form of the fourth Painlevé equation, given by

$$y'' = \frac{3}{2}\frac{y'^2}{y} - \beta y^3 - 2(z^2-\alpha)y - 4z - \frac{3}{2y}, \tag{3}$$

which is obtained from P_{IV} after the change of variables $y(z)\to 1/y(z)$.

In this paper we focus on families of semi-classical orthogonal polynomials in the so-called class one, $w'/w = C/A$, under the restrictions $\deg(A)\le 1, \deg(C)=2$, and some of their extensions, through a dependence parameter. The main goal is to obtain second order differential equations (with respect to the parameter) for the three-term recurrence relation coefficients of the orthogonal polynomials. Our approach uses a method similar as in [13]. As examples show, we recover some Painlevé equations in the well-known particular cases. Our results are illustrated by the modified Airy and Laguerre weights from [6,15,19]. We show that in these cases our general differential equations are reduced to the second and the fourth Painlevé equations. To the best of our knowledge, we believe that the results in Example 4 are new.

The paper is organised as follows. In Section 2 we present notations as well as results on semi-classical orthogonal polynomials and on discrete Painelvé equations to be used in the sequel. In Section 3 we present the main results of the paper: the derivation of two types of differential equations for the recurrence relation coefficients depending on some parameters to be specified in the text. We illustrate our results by using the well-known modified Airy and Laguerre weights from [6,15,19]. Moreover, we obtain conditions under which the general equations are reduced to the second and fourth Painlevé equations up to scaling transformations.

2. Preliminary results

2.1. Semi-classical orthogonal polynomials

Let u be a linear form defined on the space of polynomials with complex coefficients $\mathbb{P} = \text{span}\{x^k : k\in\mathbb{N}_0\}$, and let $\{P_n(x) = x^n+\dots\}$ be the sequence of monic orthogonal polynomials (SMOP) related to u, that is,

$$\langle u, P_nP_m\rangle = h_n\delta_{n,m}, \; h_n\neq 0, \quad n,m\ge 0. \tag{4}$$

It is well known that there exists a sequence of orthogonal polynomials related to u if, and only if, the moments $u_n = \langle u, x^n\rangle$, $n\ge 0$ (where we take $u_0=1$ for simplicity), satisfy $\Delta_n\neq 0$, $n\ge 0$, where Δ_n is the Hankel determinant, $\Delta_n = \det\left[u_{i+j}\right]_{i,j=0}^n$, $n\ge 0$ (see [20]). Furthermore, if $\Delta_n>0$, $n\ge 0$, then u has an integral representation in terms of a positive Borel measure, μ, supported on an infinite point set, I, of the real line

$$\langle u, x^n\rangle = \int_I x^n\, d\mu(x), \; n\ge 0, \tag{5}$$

and the orthogonality condition (4) becomes

$$\int_I P_n(x)P_m(x)d\mu(x) = h_n\delta_{n,m}, \; h_n>0, \quad n,m\ge 0.$$

In the more general case, whenever μ is an absolutely continuous measure supported on some set I, and w denotes its Radon–Nikodym derivative with respect to the Lebesgue measure, i.e. $d\mu(x) = w(x)dx$, then we will also say that $\{P_n\}$ is orthogonal with respect to the weight w.