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A high-order fully conservative block-centered finite difference method for the time-fractional advection–dispersion equation

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A B S T R A C T

Based on the weighted and shifted Grünwald–Letnikov difference operator, a new high-order block-centered finite difference method is derived for the time-fractional advection–dispersion equation by introducing an auxiliary flux variable to guarantee full mass conservation. The stability and the global convergence of the scheme are proved rigorously. Some a priori estimates of discrete norms with optimal order of convergence $O(\Delta t^3 + h^2 + k^2)$ both for solute concentration and the auxiliary flux variable are established on non-uniform rectangular grids, where Δt, h and k are the step sizes in time, space in x- and y-direction. Moreover, the applicability and accuracy of the scheme are demonstrated by numerical experiments to support our theoretical analysis.

1. Introduction

An advection–dispersion equation (ADE) has extensively been used to describe the Brownian motion of particles [20]. ADE describes the change of probability of a random function in space and time, hence it is naturally and commonly used to describe solute transport. A particle's motion has little or no spatial correlation, which is the most significant assumption underlying a Fickian diffusion process or equivalently a process of Brownian motion. Solutes moving through subsurface aquifers do not necessarily follow a Fickian process because large deviations from the stochastic process of Brownian motion are emerged due to the strong heterogeneity of the porous media [2].

It is well known that differential equations involving derivatives of non-integer order have demonstrated to be adequate models for various physical phenomena in areas like rheology, damping laws, diffusion processes, etc. At present, there is a large number of theoretical and applied works devoted to the research of fractional differential equations [3–5,12,17,24]. Recent studies show that fractional advection–diffusion equations demonstrate more effective and accurate description of the movement of solute in an aquifer than the traditional second-order advection–diffusion equations do. The fractional advection–dispersion equation is a generalization of the classical advection–dispersion equation. An Eulerian derivation of this equation has been studied by Schumer et al. [25] and it is demonstrated that highly skewed, non-Gaussian contaminant plumes with heavy leading edges can be a result of the infinite-variance particle jump distributions that arise during transport in a disordered porous medium. Zhang and his coauthors [29] developed the impact of boundary on the fractional advection–dispersion equation for solute transport in soil. The solution of its Cauchy problem in terms of the Green func-

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tions and the representations of the Green function by applying its Fourier–Laplace transforms for the space–time fractional advection–dispersion equation are derived by Huang and Liu in [7].

The numerical approximations for the fractional advection–dispersion equation are studied by many authors. Zhuang et al. [31] presented the implicit Euler approximation and some other numerical methods for the variable-order fractional advection–diffusion equation with a nonlinear source term. Effective implicit numerical methods have been demonstrated for a class of fractional advection–dispersion models in [15]. Wang and his coauthor developed a fast characteristic finite difference method in [28].

Fractional derivatives are nonlocal and they have the character of history dependence, which implies a high storage requirement. Thus, there are more and more attentions to develop high-order numerical methods which can reduce the storage requirement and computational complexity. The L_1 approximation formula to discretize the Caputo fractional derivative is modified by Gao and his coauthors and shown that the order of local truncation error is $3-\alpha$ for $0<\alpha<1$, but it is not provided a rigorous theoretical analysis about the stability and convergence of the obtained difference scheme [6]. Recently, Deng and his coauthors [9,26] have presented some high-order approximations to discretize the fractional derivatives by assembling the Grünwald–Letnikov difference operator with different weights and shifts. Following this idea, Ji and Sun [8] provided a third-order accuracy formula to approximate the time-fractional derivatives and established a compact finite difference scheme for solving the fractional sub-diffusion equation. A high-order local discontinuous Galerkin method combined with weighted and shifted Grünwald–Letnikov difference approximation is developed and discussed for a Caputo time-fractional subdiffusion equation in [14].

Block-centered finite differences, sometimes called cell-centered finite differences, can be thought as the lowest order Raviart–Thomas mixed element method [19], with proper quadrature formulation. The application of the block-centered finite difference enables us to approximate both solute concentration and the auxiliary flux with second-order accuracy on non-uniform rectangular grids to obtain the superconvergence analysis. In [1], Wheeler presents the mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences. And in 2012, a block-centered finite difference method for the Darcy–Forchheimer model is considered [22]. Besides, in [10,11,21,23] block-centered finite difference methods are developed to solve linear and nonlinear equations. Recently, a parallel CGS block-centered finite difference method for a nonlinear time-fractional parabolic equation has been studied [13].

As far as we know, there is no high-order block-centered finite difference method for the time-fractional advection–dispersion equation with the Neumann boundary condition on non-uniform rectangular grids. The main goal of this paper is to construct a high-order fully conservative block-centered finite difference method for the time-fractional advection–dispersion equation and establish the corresponding stability and error estimates. The method follows the idea of the weighted and shifted Grünwald–Letnikov difference operators [8,14,27]. By choosing shifts $(p,q,r)=(0,-1,-2)$ and utilizing the equivalence of Riemann–Liouville derivative and Caputo derivative under some regularity assumptions, a third-order accuracy formula to approximate Caputo fractional derivative is derived. Furthermore, the stable result, which just depends on initial value and source item, is derived. Besides, we demonstrate that the block centered finite difference scheme has $(\Delta t^3+h^2+k^2)$ accuracy both for the original unknown, called solute concentration, and the introduced auxiliary flux variable, in discrete L^2 norms on non-uniform rectangular grid. These error estimates are superconvergence. The key step to the superconvergence analysis, is to construct a proper relation between the auxiliary flux variable $\mathbf{q}$ and the difference of the concentration u. Then some numerical examples are carried to show the accuracy of the presented block-centered finite difference scheme.

The paper is organized as follows. In Sect. 2, we give the problem and some notations. In Sect. 3, we present the block-centered finite difference method. Then in Sect. 4, we present the analysis of stability and error estimates for the presented method. Some numerical experiments using the block-centered finite difference scheme are carried out in Sect. 5.

Through out the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

2. The problem and some notations

In this section, we first describe the problem of the time-fractional advection–dispersion equation (see [15,31]) with the Neumann condition in this paper, and present some notations which will be found helpful in the following analysis.

Find $u=u(x,y,t)$ such that

$$ {}_0^C D_t^\alpha u+\nabla\cdot(\mathbf{v}(x,y,t)u)-\nabla\cdot(\mathbf{a}(x,y,t)\nabla u)=f,\quad (x,y,t)\in\Omega\times J, $$

with Neumann boundary condition

$$ (\mathbf{v}(x,y,t)u-\mathbf{a}(x,y,t)\nabla u)\cdot\mathbf{n}=0,\quad (x,y,t)\in\partial\Omega\times J, $$

and initial condition

$$ u\,|_{t=0}=u_0(x,y),\quad (x,y)\in\Omega. $$