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Stability and convergence analysis of a Crank–Nicolson leap-frog scheme for the unsteady incompressible Navier–Stokes equations

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ABSTRACT

A fully discrete Crank–Nicolson leap-frog (CNLF) scheme is presented and studied for the nonstationary incompressible Navier–Stokes equations. The proposed scheme deals with the spatial discretization by Galerkin finite element method (FEM), treats the temporal discretization by CNLF method for the linear term and the semi-implicit method for nonlinear term. The almost unconditional stability, i.e., the time step is no more than a constant, is proven. By a new negative norm technique, the L^2-optimal error estimates with respect to temporal and spacial orientation for the velocity are derived. At last, some numerical results are provided to justify our theoretical analysis.

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1. Introduction

In this work, we consider the following unsteady incompressible Navier–Stokes equations which link the velocity field $\boldsymbol{u} = (u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t))$, the pressure $p = p(\boldsymbol{x}, t)$ and the known body force $\boldsymbol{f} = (f_1(\boldsymbol{x}, t), f_2(\boldsymbol{x}, t))$:

$$\begin{cases} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \ \operatorname{div} \boldsymbol{u} = 0, \ (\boldsymbol{x}, t) \in \Omega \times (0, T], \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \boldsymbol{x} \in \Omega; \ \boldsymbol{u}(\boldsymbol{x}, t)|_{\partial\Omega} = 0, t \in [0, T], \end{cases} \tag{1.1}$$

where $T \in (0, \infty)$, ν is the kinematic viscosity, and $\Omega \subset R^2$ is a bounded domain with a Lipschitz continuous bound $\partial\Omega$ which satisfies a further condition stated in Assumption (A1) below.

It is well known that the fully implicit time-stepping schemes are usually unconditional or almost unconditional (i.e., the time step size is less than or equal to a fixed constant) stable, however one has to deal with a nonlinear system when solving nonlinear evolution equations. In order to make the numerical method simple and easy to carry out in computer while maintaining the stability with less restrictions of time step, there appear many efficiently semi-implicit time-stepping schemes for nonlinear partial differential equations, see, for example, [3] for the Landau–Lifshitz equation, [4] for generalized Newtonian fluids, [7,9,27] for unsteady Navier–Stokes equations, [6] for a mass diffusion model, [39] for low-frequency two-fluid plasma modeling and [11,36] for nonstationary incompressible magnetohydrodynamic equations.

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The semi-implicit method is implicit with respect to linear term and semi-implicit for nonlinear term. Naturally, all these treatments lead to a linearized system so that they can save memory space (good for large scale problem) and be easy for practical implementations.

Leap-frog scheme is a popular second-order accurate difference format, and is widely used in the calculation of the atmosphere and oceans, since it preserves the wave energy conservation. Recently, more and more scholars focus on this research topic. Huang and Li, et al. [22,24,23,25,34,32,33] extended explicit/implicit leap-frog schemes for Maxwell's equations in metamaterials, a non-local dispersion model for light interaction with metallic nanostructures, or in a Cole–Cole dispersive medium, and studied their stability and convergence analysis. Usually, the standard leap-frog scheme is accompanied by other application of finite difference schemes to overcome certain unstable issues. By combination of Crank–Nicolson and leap-frog scheme, Layton and Trenchea [31] presented a time-stepping CNLF scheme for uncoupling systems of evolution equations and proved that it is conditionally stable. Since then, the reference [26], [30] and [28] promoted the proposed scheme to solve the geophysical flow, uncoupling groundwater–surface water flows, and Stokes flow plus a Coriolis term, respectively.

For the nonstationary Navier–Stokes equations, there are numerous work devoted to second-order schemes. Heywood and Rannacher [18] proved the L^2-almost unconditional convergence for the fully implicit Crank–Nicolson scheme; Shen [38] presented a second-order projection scheme, in which the viscous term and the nonlinear term were treated implicitly and the pressure term explicitly; He [8,12] proposed Crank–Nicolson extrapolation scheme based on FEM and proved its almost unconditional stability and optimal error estimates; Later, He and Sun [13] investigated the scheme based on stabilized FEM and proved its unconditional stability; Chan, He, Zhang et al. [2] derived the H^1-unconditional convergence for the first- and second-order semi-implicit schemes.

In this article, inspired by [31], we present a fully discrete CNLF semi-implicit scheme based on FEM for numerically solving the unsteady incompressible Navier–Stokes equations. The almost unconditional stability for the proposed scheme is derived. We divide the error into two part, one is the differences between exact solution $\boldsymbol{u}(t)$ and the finite element semi-discrete solution $\boldsymbol{u}_h(t)$, the other between $\boldsymbol{u}_h(t_n)$ and the fully discrete approximation solution $\boldsymbol{u}_h^n$, where $t_n = n\tau$ with equal time step size τ. The main conclusions of this paper is that the L^2-optimal convergence rate of second order in time between $\boldsymbol{u}_h(t_n)$ and $\boldsymbol{u}_h^n$ is proven by negative norm technique. The negative norm skill plays an important role in obtaining the optimal error estimate in time in the recent work, for example, the unconditional convergence of the Euler semi-implicit scheme for the 3D incompressible magnetohydrodynamic flow [11], the almost unconditional convergence of first-order decoupled FEM [15] and conditional convergence of a second-order decoupled implicit/explicit FEM [14] of the 3D primitive equations of the ocean.

The rest of this paper is organized as follows. In Section 2, a functional setting of the problem (1.1) is given. Subsequent Galerkin FEM and some regularity of its solution are shown. Then the fully discrete CNLF semi-implicit scheme are presented. Section 3 is devoted to the almost unconditional stability analysis of the proposed scheme. In Section 4, the L^2-optimal error estimate is proven. Some numerical experiments are implemented to confirm the theoretical analysis in last section.

2. Fully discrete CNLF scheme for Navier–Stokes equations

In this section, abstract functional setting of Navier–Stokes equations is firstly given. The Galerkin mixed finite element formula is provided. Lastly, the fully discrete CNLF scheme for the studying equations is presented based on mixed FEM.

As usual, we employ the standard notation of vector Sobolev spaces.

$$\boldsymbol{X} = \boldsymbol{H}_0^1(\Omega) = (H_0^1(\Omega))^2, \quad \boldsymbol{X}_0 = \left\{\boldsymbol{v} \in \boldsymbol{X} : \operatorname{div} \boldsymbol{v} = 0\right\},$$
$$M = L_0^2(\Omega) = \left\{q \in L^2(\Omega) : \int_\Omega q d\boldsymbol{x} = 0\right\}.$$

The space $L^2(\Omega)$ is equipped with the usual L^2-scalar product $(\cdot, \cdot)$ and L^2-norm $\|\cdot\|_{L^2}$ or $\|\cdot\|_0$. The space $\boldsymbol{H}^1(\Omega)$ is equipped with norm $\|\boldsymbol{u}\|_{\boldsymbol{H}_0^1}$ or semi-norm $\|\nabla \boldsymbol{u}\|_{\boldsymbol{L}^2}$. We denote by $\boldsymbol{H}$ the closed subset of $\boldsymbol{L}^2(\Omega)$, i.e.,

$$\boldsymbol{H} = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \operatorname{div} \boldsymbol{v} = 0, \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\}.$$

Furthermore, in the rest of the paper, we frequently adopt

$$a(\boldsymbol{u}, \boldsymbol{v}) = \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}), \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X},$$
$$d(\boldsymbol{v}, q) = (\operatorname{div} \boldsymbol{v}, q), \ \forall \boldsymbol{v} \in \boldsymbol{X}, \ \forall q \in M,$$

and the explicitly skew-symmetric convection term

$$b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}) + \frac{1}{2}((\operatorname{div} \boldsymbol{u})\boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2}(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v}), \ \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}.$$

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