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Splitting schemes for unsteady problems involving the grad-div operator

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ABSTRACT

In this paper we consider various splitting schemes for unsteady problems containing the grad-div operator. The fully implicit discretization of such problems would yield at each time step a linear problem that couples all components of the solution vector. In this paper we discuss various possibilities to decouple the equations for the different components that result in unconditionally stable schemes. If the spatial discretization uses Cartesian grids, the resulting schemes are Locally One Dimensional (LOD). The stability analysis of these schemes is based on the general stability theory of additive operator-difference schemes developed by Samarskii and his collaborators. The results of the theoretical analysis are illustrated on a 2D numerical example with a smooth manufactured solution.

1. Introduction

The grad-div ($\nabla\nabla\cdot$) operator appears in various problems in science and engineering, the obvious examples being the Navier–Stokes equations in the so-called stress-divergence form (see e.g. [4]), the equations of linear elasticity, some formulations of the Maxwell equations, the so-called Galbrun equation in aero-acoustics, etc. (see e.g. [7,8]). For a more detailed list of applications containing this operator the reader is referred to [1]. Sometimes, this operator may appear as a result of a numerical regularization of problems involving incompressible fields as in the case of the so-called artificial compressibility methods (see for example [15], [5], and [6]).

The main issue with the presence of a $\nabla\nabla\cdot$ operator is that in implicit discretizations it couples the equations for the various components of the vector field, similarly to the rot-rot ($\nabla\times\nabla\times$) operator. If this coupling can be avoided, the resulting linear system would be easier to solve, no matter if direct or iterative linear solvers are used. Such decoupling of the vectorial problem has already been proposed, for the case of the Maxwell equations involving the $\nabla\times\nabla\times$ operator, in [16] and [17], section 3.4. It results in a set of uncoupled elliptic problems for each component of the solution vector. In the present paper we show that the same approach works in the case of problems involving $\nabla\nabla\cdot$ operators. If the spatial discretization is performed on Cartesian grids, then these splittings result in Locally One Dimensional (LOD) schemes that are very efficient from computational standpoint. Note that while the $\nabla\times\nabla\times$ operator is positive the $\nabla\nabla\cdot$ operator is only non-negative and this difference requires some modifications in the stability analysis of the resulting schemes. Nevertheless, the stability analysis of the schemes proposed in this paper is based on the general stability theory of additive operator-

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difference schemes developed by Samarskii and his collaborators (see [12,13]). Taking into account the non-negativity of the $\nabla\nabla\cdot$ operator, we prove the unconditional stability and provide *a priori* estimates for several decoupling schemes that have been used for vectorial problems with positive operators (see [16] and [17], section 3.4).

The rest of the paper is organized as follows. In the next section we formulate the problem and derive some *a priori* estimates for it. In section 3 we consider some standard two-level schemes for unsteady problems with grad-div operators, and discuss their stability. In section 4 two-level splitting schemes with implicit block-diagonal or block-triangular structure are considered, as well as alternating block-triangular two- and three-level schemes. The theoretical results are verified on a numerical example with a manufactured solution in section 5. Finally, in the last section we summarize the results of this paper.

2. Problem formulation

Consider the problem: Find $\boldsymbol{u}(\boldsymbol{x},t) = (u_1, ..., u_d)^T$ that satisfies:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \operatorname{grad}(k(\boldsymbol{x}) \operatorname{div} \boldsymbol{u}) = \mathbf{f}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T, \tag{1a}$$

$$(\boldsymbol{u} \cdot \boldsymbol{n}) = 0, \quad \boldsymbol{x} \in \partial\Omega, \tag{1b}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1c}$$

where $\Omega \subset R^d$, $d = 2,3$ is a bounded polygonal domain with a Lipschitz continuous boundary $\partial\Omega$, and $\boldsymbol{n}$ is the outward normal to $\partial\Omega$. The coefficient $k(\boldsymbol{x}) \ge 0$ and the source term $f(\boldsymbol{x},t)$ are supposed to be sufficiently smooth.

Remark 1. We chose here to study the case of parabolic equations involving the grad-div operator. However, the schemes and their stability analysis extend in a straightforward manner to the case of second-order in time equations. For example, consider the so-called Galbrun equation (see e.g. [2], [3]):

$$\frac{D^2 \boldsymbol{u}}{Dt^2} - \operatorname{grad}(k(\boldsymbol{x}) \operatorname{div} \boldsymbol{u}) = \mathbf{f}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \Omega, \quad 0 < t \le T, \tag{2}$$

where $\displaystyle\frac{D\boldsymbol{u}}{Dt} = \frac{\partial \boldsymbol{u}}{\partial t} + M(y)\frac{\partial \boldsymbol{u}}{\partial x}$.

Let $(\cdot,\cdot)$ denote the standard L^2 inner product over Ω, and $\|\cdot\|$ be the corresponding norm for scalar and vector functions $v(\boldsymbol{x})$ and $\boldsymbol{v}(\boldsymbol{x})$, correspondingly:

$$(v,w) = \int_\Omega v(\boldsymbol{x}) w(\boldsymbol{x}) d\boldsymbol{x}, \quad \|v\| = (v,v)^{1/2},$$

$$(\boldsymbol{v},\boldsymbol{w}) = \sum_{i=1}^{d} (v_i, w_i), \quad \|\boldsymbol{v}\| = \Big(\sum_{i=1}^{d} \|v_i\|^2\Big)^{1/2}.$$

It is quite obvious that problem (1) is well posed and we can easily obtain *a priori* estimates for it. To reduce the complexity of the notation in the paper we use calligraphic letters for denoting operators in infinite dimensional spaces and standard capital letters for their finite dimensional approximations. Equation (1a) on the set of functions satisfying (1b) can be reformulated as a standard Cauchy problem for a first order evolutionary equation:

$$\frac{d\boldsymbol{u}}{dt} + \mathcal{A}\boldsymbol{u} = \boldsymbol{f}(t), \quad 0 < t \le T, \tag{3a}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{3b}$$

where $\boldsymbol{u}(t) = \boldsymbol{u}(\cdot,t)$. It is clear that $\mathcal{A}$ is self-adjoint and non-negative in $L^2(\Omega)$ i.e.:

$$\mathcal{A} = \mathcal{A}^* \ge 0. \tag{4}$$

To obtain an *a priori* estimate for the solution of (3), (4), we multiply (3a) by $\boldsymbol{u}$ to obtain:

$$\left(\frac{d\boldsymbol{u}}{dt}, \boldsymbol{u}\right) + (\mathcal{A}\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u}).$$

Taking into account (4) and the identity

$$\left(\frac{d\boldsymbol{u}}{dt}, \boldsymbol{u}\right) = \frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}\|^2 = \|\boldsymbol{u}\|\frac{d}{dt}\|\boldsymbol{u}\|$$