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Some distance magic graphs

Aloysius Godinho*, T. Singh

Department of Mathematics, Birla Institute of Technology and Science Pilani, K K Birla, Goa Campus, NH-17B, Zuarinagar, Goa, India

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Abstract

A graph $G = (V, E)$, where $|V| = n$ and $|E| = m$ is said to be a *distance magic graph* if there exists a bijection from the vertex set V to the set $\{1, 2, \dots, n\}$ such that, $\sum_{v \in N(u)} f(v) = k$, for all $u \in V$, which is a constant and independent of u, where $N(u)$ is the open neighborhood of the vertex u. The constant k is called the distance magic constant of the graph G and such a labeling f is called distance magic labeling of G. In this paper, we present new results on distance magic labeling of C_n^r and neighborhood expansion $D_n(G)$ of a graph G.

Keywords: Distance magic labeling; Sigma labeling; Circulant graphs

1. Introduction

For standard terminology and notation in graph theory we follow Harary [1]. Also, unless mentioned otherwise, all graphs considered here are simple, finite, without loops and multiple edges.

The concept of distance magic labeling of a graph was motivated by the construction of magic squares. Suppose, we have a magic square consisting of n rows and m columns and each row sum is k. Consider a complete multipartite graph with each row of the magic square representing a partite set and we label each vertex with the corresponding integer in the magic square. We find that the sum of the labels of all the vertices at distance 1 (i.e., an open neighborhood set of vertices) for each vertex is the same and is equal to $(n - 1)k$.

Let $G = (V, E)$ be a graph with n vertices and m edges. Let f be a bijection from the vertex set $V(G)$ onto the set $\{1, 2, \dots, n\}$ such that for every vertex u, $w(u) = \sum_{uv \in E(G)} f(v) = k$ which is a constant and independent of u. This constant value k is called the *distance magic constant* of the graph G, then f is called a *distance magic labeling* of G and the graph which admits such a labeling is called a *distance magic graph*.

The concept was introduced by Vilfred [2] as *sigma labeling* of graph and further developed by Jinnah [3] and Vilfred [4] (see also, [5,6]). This concept was independently studied by Simanjuntak, Rodgers and Miller [7] who used the terminology 1-*vertex magic vertex labeling*. The same concept was introduced independently in more general

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* Corresponding author.
E-mail addresses: p20140001@goa.bits-pilani.ac.in (A. Godinho), tksingh@goa.bits-pilani.ac.in (T. Singh).

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way as *neighborhood magic labeling* by Acharya et al. [8]. A survey on distance magic labelings is given in [6]. The following results are known in the literature (see also [9]).

Theorem 1.1 ([4]). *Let G be a distance magic graph of order n with magic constant k. Let $f : V(G) \to \{1, 2, \dots, n\}$ be a distance magic labeling of G. Let $d(u_i)$ denote the degree of the vertex u_i, then $\sum_{i=1}^{n} f(u_i)d(u_i) = nk$.*

Corollary 1.2 ([4]). *Let G be a r-regular distance magic graph on n vertices. Then the distance magic constant $k = \frac{r(n+1)}{2}$.*

Corollary 1.3 ([4]). *No r-regular graph with r odd can be a distance magic graph.*

Theorem 1.4 ([8]). *Every graph is a subgraph of a distance magic graph.*

Theorem 1.5 ([8]). *The graph $P_n \Box C_k$ is not a distance magic graph when n is odd.*

Theorem 1.6 ([10]). *$C_n \Box C_k$ is distance magic if and only if $n = k \equiv 2 (\text{mod } 4)$.*

In [10], Rao et al. posed the following problems:

Problem 1.7. Characterize 4-regular distance magic graphs.

Problem 1.8. Find all distance magic labelings of $C_n \Box C_k$, $n = k \equiv 2 (\text{mod } 4)$.

Fronćek et al. [11] proved the following result on 4-regular distance magic graphs:

Theorem 1.9. *There exists a 4-regular distance magic graph on an odd number of vertices n if and only if $n \geq 17$.*

Let n and r be positive integers such that $n \geq 3$ and $r \leq \frac{n-1}{2}$. The graph C_n^r is a graph on n vertices $\{u_0, u_1, \dots, u_{n-1}\}$ with the edge set $E(C_n^r) = \{u_i u_{i+j} : 0 \leq i \leq n-1,\ 1 \leq j \leq r\}$. The index i in u_i is assumed to be taken modulo n. Notice from the above definition that the graph C_n^r is $2r$-regular. The problem of obtaining necessary and sufficient conditions for existence of distance magic labeling for the graph C_n^r has been studied by Cichacz [12]. She proved the following results.

Theorem 1.10. *If r is odd, the graph C_n^r is distance magic if and only if $2r(r+1) \equiv 0 (\text{mod } n)$, $n \geq 2r+2$ and $\frac{n}{gcd(n,r+1)} \equiv 0 (\text{mod } 2)$.*

Theorem 1.11. *If C_n^r is distance magic then n is even.*

In this paper we shall discuss the distance magic labeling of C_n^r when r is even and the neighborhood expansion $D_n(G)$ of the graph G.

2. Distance magic labeling of C_n^r

We denote the vertex set of C_n^r by $V(C_n^r) = \{u_0, u_1, \dots, u_{n-1}\}$, such that the vertex u_i is adjacent to the set of vertices $\{u_{i+j}, u_{i-j} : 1 \leq j \leq r\}$. For a bijection $f : V(C_n^r) \to \{1, 2, \dots, n\}$, we shall use the notation f_i to denote the label of u_i i.e $f(u_i) = f_i$. The index i in u_i and f_i are assumed to be taken modulo n.

Suppose the graph C_n^r is distance magic. From Corollary 1.2 it follows that the distance magic constant of C_n^r is $r(n+1)$. Hence if f is a distance magic labeling and $u_i \in V$, the weight of u_i is $w(u_i) = \sum_{j=1}^{r}(f_{i+j} + f_{i-j}) = r(n+1)$.

Lemma 2.1. *If C_n^r is distance magic, then for any $u_i \in V(C_n^r)$ and $\lambda \in \mathbb{Z}$,*

$$f_i + f_{i+r+1} = f_{i+\lambda r} + f_{i+(\lambda+1)r+1}. \tag{1}$$