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Structure of edges in a 4-connected graph not contained in triangles and the number of contractible edges

Yoshimi Egawa[a], Keiko Kotani[b], Shunsuke Nakamura[a,*]

[a] *Department of Applied Mathematics, Tokyo University of Science, Shinjuku-ku, Tokyo, 162-8601, Japan*
[b] *Department of Mathematics, Tokyo University of Science, Shinjuku-ku, Tokyo, 162-8601, Japan*

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Abstract

Let G be a 4-connected graph, and let $\tilde{E}(G)$ denote the set of those edges of G which are not contained in a triangle, and let $E_c(G)$ denote the set of 4-contractible edges of G. We show that if $3 \leq |\tilde{E}(G)| \leq 4$ or $|\tilde{E}(G)| \geq 7$, then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$ unless G has one of the three specified configurations.
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Keywords: 4-connected graph; Contractible edge; Triangle

1. Introduction

In this paper, we consider only finite undirected simple graphs with no loops and no multiple edges.

Let $G = (V(G), E(G))$ be a graph. A subset S of $V(G)$ is called a cutset if $G - S$ is disconnected. A cutset with cardinality i is simply referred to as an i-cutset. For an integer $k \geq 1$, we say that G is k-connected if $|V(G)| \geq k + 1$ and G has no $(k - 1)$-cutset. For $x \in V(G)$, $N_G(x)$ denotes the neighborhood of x and $\deg_G(x)$ denotes the degree of x; thus $\deg_G(x) = |N_G(x)|$. For $e \in E(G)$, we let $V(e)$ denote the set of endvertices of e. The complete graph of order n is denoted by K_n. For a graph H, let nH denote the graph with n components, each isomorphic to H.

Let G be a 4-connected graph. A 4-cutset S of G is said to be trivial if there exists a vertex z of degree 4 such that $N_G(z) = S$; otherwise it is said to be nontrivial. For $e \in E(G)$, we let G/e denote the graph obtained from G by contracting e into one vertex (and replacing each resulting pair of double edges by a simple edge). We say that e is 4-contractible or 4-noncontractible according as G/e is 4-connected or not. A 4-noncontractible edge $e = ab$ is said to be trivially 4-noncontractible if there exists a vertex z of degree 4 such that $za, zb \in E(G)$. We let $E_c(G)$, $E_n(G)$ and $E_{tn}(G)$ denote the set of 4-contractible edges, the set of 4-noncontractible edges and the set of trivially

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* Corresponding author.
E-mail addresses: kkotani@rs.kagu.tus.ac.jp (K. Kotani), 1417701@ed.tus.ac.jp (S. Nakamura).

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4-noncontractible edges, respectively. Note that if $|V(G)| \geq 6$, then $e \in E_n(G)$ if and only if there exists a 4-cutset S such that $V(e) \subseteq S$. Finally we let $\tilde{E}(G)$ denote the set of those edges of G which are not contained in a triangle.

The following result concerning a lower bound on $|E_c(G)|$ in terms of $|\tilde{E}(G)|$ was proved by Ando and Egawa in [1].

Theorem A. *Let G be a 4-connected graph, and suppose that $|\tilde{E}(G)| \geq 15$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$.*

In [1], it was also shown that the lower bound on $|E_c(G)|$ in Theorem A is best possible. The following similar result concerning $|E_c(G)|$ and $|\tilde{E}(G)|$ was proved in [2].

Theorem B. *Let G be a 4-connected graph, and suppose that $7 \leq |\tilde{E}(G)| \leq 8$ or $10 \leq |\tilde{E}(G)| \leq 14$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$.*

Note that the case where $|\tilde{E}(G)| = 9$ is missing in Theorems A and B. In this paper, we take up the case where $|\tilde{E}(G)| = 9$, and show that the same conclusion as in Theorems A and B holds unless G has a specified configuration. We also consider the cases where $|\tilde{E}(G)| = 3$ and 4, and prove similar results.

In order to state our results, we need some more definitions. Let G be a 4-connected graph. Let $\tilde{V}$ denote the set of those vertices of G which are incident with an edge in $\tilde{E}(G) \cap E_n(G)$, and let $\hat{G}$ denote the subgraph of G induced by the edge set $\tilde{E}(G) \cap E_n(G)$; that is to say, $\tilde{V} = \cup_{e \in \tilde{E}(G) \cap E_n(G)} V(e)$ and $\hat{G} = (\tilde{V}, \tilde{E}(G) \cap E_n(G))$. We let Y and Y^* denote the graphs of order 6 defined by

$$V(Y) = \{w, z\} \cup \{v_i \mid 1 \leq i \leq 4\}, \quad E(Y) = \{wz, v_1w, v_2w, v_3w, v_4z\},$$
$$V(Y^*) = V(Y), \quad E(Y^*) = \{wz, v_1w, v_2w, v_3z, v_4z\}.$$

Our main result is the following theorem.

Theorem 1. *Let G be a 4-connected graph, and suppose that $3 \leq |\tilde{E}(G)| \leq 4$ or $|\tilde{E}(G)| = 9$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 4)/4$. Further we have $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$ unless one of the following holds:*
(1) *$|\tilde{E}(G)| = 3$ and $\hat{G} = K_2$;*
(2) *$|\tilde{E}(G)| = 4$ and $\hat{G} = 2K_2$; or*
(3) *$|\tilde{E}(G)| = 9$ and $\hat{G} = Y^*$.*

We remark that, as we will show in Section 6 by constructing examples, the structures of $\hat{G}$ described (1) through (3) of Theorem 1 are indeed exceptional; that is to say, there exist graphs G such that $\hat{G}$ is as in (1) through (3) of Theorem 1 and G does not satisfy the inequality $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$. In this paper, we derive Theorem 1 from the following theorem.

Theorem 2. *Let G be a 4-connected graph, and suppose that $1 \leq |E(\hat{G})| \leq 3$ or $|E(\hat{G})| = 5$. Then $|E_c(G)| \geq (|E(\hat{G})| + 2)/2$.*
Further we have $|E_c(G)| \geq (|E(\hat{G})| + 4)/2$ unless one of the following holds:
(1) *$|E(\hat{G})| = 1$ or 3;*
(2) *$|E(\hat{G})| = 2$ and $\hat{G} = 2K_2$; or*
(3) *$|E(\hat{G})| = 5$ and $\hat{G} = Y$ or Y^*.*

Combining Theorems A, B and 1, we obtain the following theorem.

Theorem 3. *Let G be a 4-connected graph, and suppose that $3 \leq |\tilde{E}(G)| \leq 4$ or $|\tilde{E}(G)| \geq 7$. Then $|E_c(G)| \geq (|\tilde{E}(G)| + 4)/4$. Further we have $|E_c(G)| \geq (|\tilde{E}(G)| + 8)/4$ unless one of conditions* (1) *through* (3) *in the statement of Theorem 1 holds.*

In passing, we here include comments concerning the cases not considered in Theorem 3. As for the case where $|\tilde{E}(G)| = 1$, it was shown in [3] that unless $\hat{G} = K_2$, we have $|E_c(G)| \geq 3$ $(= \lceil (|\tilde{E}(G)| + 8)/4 \rceil)$. As for the cases where $|\tilde{E}(G)| = 2$, 5 and 6, the complete determination of exceptional structure of $\hat{G}$ seems rather involved, and we are hoping to discuss these three cases in a subsequent paper.

Our notation is standard, and is mostly taken from Diestel [4]. Possible exceptions are as follows. For a graph G, let $w(G)$ be the number of components of G. For two graphs G and G' with $V(G) \cap V(G') = \emptyset$, let $G \cup G'$ denote a graph

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