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Client–server and cost effective sets in graphs

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Abstract

For any integer $k \geq 0$, a set of vertices S of a graph $G = (V, E)$ is *k-cost-effective* if for every $v \in S$, $|N(v) \cap (V \setminus S)| \geq |N(v) \cap S| + k$. In this paper we study the minimum cardinality of a maximal k-cost-effective set and the maximum cardinality of a k-cost-effective set. We obtain Gallai-type results involving the k-cost-effective and global k-offensive alliance parameters, and we provide bounds on the maximum k-cost-effective number. Finally, we consider k-cost-effective sets that are also dominating. We show that computing the k-cost-effective domination number is NP-complete for bipartite graphs. Moreover, we note that not all trees have a k-cost-effective dominating set and give a constructive characterization of those that do.

Keywords: Cost effective sets; k-cost-effective sets; Domination; k-cost effective domination; Global k-offensive alliances

1. Introduction

In a graph model of a computer network, specific vertices act as servers, in that they provide to neighboring vertices various computing facilities, such as data bases and specialized software. Ideally, for economical reasons, each server serves as many vertices (non-servers, or clients) as possible. Thus, in general, we would seek to establish a set of servers, each of which is serving a maximal number of clients.

In this paper, we introduce a generalization of cost effective sets suggested by Hedetniemi, Hedetniemi, Kennedy, and McRae [1]. As an introduction and motivation, in Section 2, we set up several different client–server models and objectives. The remainder of the paper will focus on the generalization of cost effective sets, namely, k-cost-effective sets (defined in Section 2).

We consider finite, undirected, and simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. We shall use the following terminology. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *degree* of v, denoted by $\deg_G(v)$, is the cardinality

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of its open neighborhood. We denote by $\Delta(G) = \Delta$ and $\delta(G) = \delta$ the *maximum degree* and the *minimum degree* of a vertex in $V(G)$, respectively. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. The *open neighborhood of a set* $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood of a set* S is the set $N[S] = N(S) \cup S = \cup_{v \in S} N[v]$. For any $S \subseteq V$, we denote the subgraph of G induced by S as $G[S]$.

A set $S \subset V$ is called *independent* if no two vertices in S are adjacent. The *independent domination number* $i(G)$ is the minimum cardinality of a maximal independent set, and the *independence number* $\beta(G)$ equals the maximum cardinality of an independent set in G. A set $S \subset V$ is called a *dominating set* if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ equals the minimum cardinality of a dominating set in G. In what follows, for any parameter $\mu(G)$ associated with a graph property $\mathcal{P}$, we refer to a set of vertices with property $\mathcal{P}$ and cardinality $\mu(G)$ as a $\mu(G)$-*set*.

A *Gallai Theorem* is a result of the form: $\alpha(G) + \lambda(G) = n$, where G is a graph of order $n = |V|$ and $\alpha(G)$ and $\lambda(G)$ are two, non-negative integer-valued parameters of a graph. An example of a Gallai Theorem follows.

A set $S \subset V$ is called *enclaveless* if no vertex $u \in S$ satisfies $N[u] \subseteq S$, that is, every vertex $u \in S$ has at least one neighbor in $V \setminus S$. The *enclaveless number* $\Psi(G)$ equals the maximum cardinality of an enclaveless set in G. The following result is found in [2].

Proposition 1. *For any graph G of order n, $\gamma(G) + \Psi(G) = n$.*

In Section 3, we obtain Gallai theorems involving k-cost-effective numbers, and in Section 4, we provide bounds on the maximum k-cost-effective number. Finally, in Section 5, we consider k-cost-effective sets that are also dominating. We show that computing the k-cost-effective domination number is NP-complete for bipartite graphs. Moreover, we give a constructive characterization of the trees having a k-cost-effective dominating set, since not all trees have such sets.

We conclude this section by mentioning a generalization of independence and domination that will be useful in our results. In [3,4], Fink and Jacobson introduced the concepts of p-domination and p-dependence. Let p be a positive integer. A subset S of V is a *p-dominating set* of G if for every vertex $v \in V \setminus S$, $|N(v) \cap S| \geq p$. A *p-dependent set* is a subset D of V such that the maximum degree in the subgraph $G[D]$ induced by the vertices of D is at most $p-1$. The *p-domination number* $\gamma_p(G)$ is the minimum cardinality of a p-dominating set of G, and the *p-dependence number* $\beta_p(G)$ is the maximum cardinality of a p-dependent set of G. Notice that a 1-dominating set (respectively, a 1-dependent set) is a dominating set (respectively, an independent set), and so $\gamma(G) = \gamma_1(G)$ and $\beta_1(G) = \beta(G)$. For more information on k-domination and k-dependence, see the survey by Chellali et al. [5].

2. Client–server models

In this section, we describe several client–server models, concluding with the definition of k-cost-effectiveness, the focus of this paper.

Differential sets

In this model, we seek a set of servers that collectively serve a maximum number of clients minus servers. The following definitions were introduced by Haynes et al. in 2006 [6]. The *differential of a set* $S \subset V$ is defined as $\partial(S) = |N(S)| - |S|$, while the *differential of a graph* G is defined as $\partial(G) = max\{\partial(S) : S \subset V\}$. Thus, it is apparent that a set S of servers that maximizes the number of clients served minus the number of servers served is just a set that defines the differential of a graph.

Client number

In the next model, we seek to find a set of servers that collectively serves a maximum number of clients. Suppose we define the *client number* $CN(G)$ to equal the maximum number of clients that can be served by a set S of servers, that is, $CN(G) = max\{|N(S) \cap (V \setminus S)| : S \subset V\}$. As it turns out, it is easy to show the following; we leave the proof to the interested reader.

Proposition 2. *For any graph G of order n, $\gamma(G) + CN(G) = n$.*

Corollary 3. *For any graph G, $\Psi(G) = CN(G)$.*