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Zeta functions from graphs

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Abstract

The location of the nontrivial poles of a generalized zeta function is derived from the spectrum of Ramanujan graphs and bounds are established for irregular graphs. The existence of a similarity transformation of the diagonal matrix given by a specified set of eigenvalues to an adjacency matrix of a graph is proven, and the method yields a set of finite graphs with eigenvalues determined approximately by a finite subset of the poles of the Ihara zeta function.

Keywords: Graph; Zeta function; Adjacency matrix; Eigenvalues

1. Introduction

The class of Ramanujan graphs has symmetries which affect the spectrum of the eigenvalues. The spectrum is required to satisfy a set of inequalities. The zeta function will be demonstrated to satisfy the inequalities, and therefore, symmetries may be used to establish the location of the nontrivial zeros.

The inverse of the Ihara zeta function is defined to be equal to the product of a polynomial factor and a determinant depending on the eigenvalues of the adjacency matrix of a graph. The eigenvalues will be defined by a relation with the prime numbers and proven to satisfy the condition for a Ramanujan graph. A similarity transformation is required to transform the diagonal matrix of eigenvalues to a matrix with integer entries. This new adjacency matrix generally would be irregular because the elements can be unequal. Positivity can be verified for a set of eigenvalues sufficiently bounded away from zero. The Ihara zeta function of an irregular adjacency matrix with eigenvalues satisfying the bound $\lambda_n \leq 2\sqrt{q}$ will have nontrivial poles in the region $0 \leq Re\ s \leq 1$. It will follow that a set of eigenvalues may be chosen such that the Ihara zeta function is the product of a multiplicative factor and the product that may be equated to a generalized zeta function. Since the multiplicative factor only could have a zero at $s = 0$ if the minimum degree is larger than 2, the nontrivial poles of the generalized zeta function would be located in $\{0 < Re\ s < 1\}$.

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2. The existence of Ramanujan graphs with the eigenvalue spectrum of the Riemann zeta function

The Ihara zeta function of a $q+1$-regular graph X [1,2] is

$$Z_X(u) = \prod_{[C]\ primitive} (1 - u^{\nu(C)})^{-1} \tag{2.1}$$

where the equivalence class $[C]$ is defined by $C = (v_1, \ldots, v_m = v_1) \sim D = (w_1, \ldots, w_m = w_1)$ if $w_j = v_{j+k}$ and $\nu(C)$ is the length of C. If X is a connected $(q+1)$-regular graph with adjacency matrix A and r is the rank of the fundamental group, defined by homotopy classes of cycles in X, the determinant formula [1,3] for the Ihara zeta function is

$$Z_X^{-1}(u) = (1-u^2)^{r-1}\, det(I - Au + qu^2 I). \tag{2.2}$$

If the spectrum of A is the set of eigenvalues $\{\lambda_n\}$

$$Z_X^{-1}(u) = (1-u^2)^{r-1} \prod_n (1 - \lambda_n u + qu^2). \tag{2.3}$$

Ramanujan graphs have eigenvalues λ_n of the adjacency matrix which satisfy $\lambda_n \le 2\sqrt{q}$ if $|\lambda_n| \ne q+1$. It has been demonstrated that $Z_X(q^{-s})$ satisfies an analogue of the Riemann hypothesis if and only if X is a Ramanujan graph [1].

Theorem. *The approximate location of the nontrivial poles of a generalized zeta function*

$$\prod_n \left(1 - \frac{1}{p_n^s} - \tilde{c}_{4n} + \frac{1}{\omega_X^{2s-1}}\right)^{-1}$$

where $\omega_X = \frac{1}{R_X}$, R_X *is the radius of the convergence of the Ihara zeta function* $Z_X(u)$ *of a graph* X *with maximum degree* $q+1$ *and minimum degree* $p+1$, *with* $p \le \omega_X \le q$, *and*

$$\lim_{N\to\infty} \sum_{\substack{n \\ p_n \le N}} \tilde{c}_{4n} = \lim_{N\to\infty} \left[\ln\ln 2 - 1 - \int_2^\infty \frac{\mathcal{R}(t)}{t\,(\ln t)^2} dt - \ln\ln N - \mathcal{O}\left(\frac{1}{\ln N}\right)\right],$$

with $\mathcal{R}(t) = \mathcal{O}(1)$, *in the region* $0 \le Re\ s \le 1$, *and the absence of nonreal poles when* $\frac{1}{2}\frac{\ln p}{\ln \omega_X} < Re\ s < \frac{1}{2}\frac{\ln q}{\ln \omega_X}$, *follows from the analogue of the Riemann hypothesis for* $Z_X(u)$, *when there exists a similarity transformation from the diagonal matrix with these eigenvalues to matrix with nearly integer values that can be rounded to an integer-valued adjacency matrix.*

Proof. The graphs also can be viewed as triangulations of a Riemann surface. The nearest neighbours of a site define correlations which are step function versions of a distributed function that varies inversely with respect to the distance. The Green function is the inverse of the Laplacian, and the eigenvalues of the adjacency matrix are related to the eigenvalues of the Laplacian on the Riemann surface. As the genus of the surface increases, the order of the adjacency matrix tends to infinity. If the inverse of the Ihara zeta function coincides with the product of $(1-u^2)^{r-1}$ and the inverse Riemann zeta function, then the Riemann hypothesis is proven provided that the graph is Ramanujan. A connection between the Riemann hypothesis for the zeta function $\zeta(s)$ and the Ihara zeta function of graphs $Z_X(u)$ has been developed for finite graphs [4].

Suppose that $q = 1$ and consider the equality

$$\rho_n u - u^2 = \frac{1}{p_n^s} \tag{2.4}$$

where p_n is the nth prime, beginning with $p_1 = 2$, and ρ_n is a variable that can be identified with the eigenvalue λ_n if the inequality for the Ramanujan graph is satisfied. This equation can be satisfied for example by

$$\begin{aligned} \rho_n &= 1 + \frac{1}{p_n^s} \\ u &= 1. \end{aligned} \tag{2.5}$$

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