ARTICLE IN PRESS

Available online at www.sciencedirect.com
ScienceDirect

AKCE International Journal of Graphs and Combinatorics ■ (■■■■) ■■■–■■■

AKCE International Journal of Graphs and Combinatorics

www.elsevier.com/locate/akcej

Results on Laplacian spectra of graphs with pockets

Sasmita Barik*, Gopinath Sahoo

School of Basic Sciences, IIT Bhubaneswar, Bhubaneswar, 751007, India

Received 1 January 2017; received in revised form 21 November 2017; accepted 21 November 2017
Available online xxxx

Abstract

Let F, H_v be simple connected graphs on n and $m+1$ vertices, respectively. Let v be a specified vertex of H_v and $u_1, \ldots, u_k \in F$. Then the graph $G = G[F, u_1, \ldots, u_k, H_v]$ obtained by taking one copy of F and k copies of H_v, and then attaching the ith copy of H_v to the vertex u_i, $i = 1, \ldots, k$, at the vertex v of H_v (identify u_i with the vertex v of the ith copy) is called a graph with k pockets. In 2008, Barik raised the question that 'how far can the Laplacian spectrum of G be described by using the Laplacian spectra of F and H_v?' and discussed the case when $\deg(v) = m$ in H_v. In this article, we study the problem for more general cases and describe the Laplacian spectrum. As an application, we construct new nonisomorphic Laplacian cospectral graphs from the known ones.

Keywords: Laplacian matrix; Laplacian spectrum; Join; Pockets

1. Introduction

Throughout this article we consider only simple graphs. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G)$. The *adjacency matrix* of G, denoted by $A(G)$, is the $n \times n$ matrix whose (i, j)-th entry is 1, if i and j are adjacent in G and 0, otherwise. The *Laplacian matrix* of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal degree matrix of G. It is well known that $L(G)$ is a symmetric positive semidefinite matrix. Throughout this paper the *Laplacian spectrum* of G is defined as $\sigma(G) = (\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G))$, where $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G)$ are the eigenvalues of $L(G)$ arranged in nondecreasing order. For any graph G, $\lambda_1(G) = 0$ and is afforded by the all ones eigenvector $\mathbf{1}$. There is an extensive literature on works related to Laplacian matrices and their spectra. Interested readers are referred to [1,2] and the references therein.

Two graphs are said to be *Laplacian cospectral* if they share same Laplacian spectrum. Haemers and Spence [3] enumerated the numbers for which there are at least two graphs with the same Laplacian spectrum and gave some techniques for their construction.

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: sasmita@iitbbs.ac.in (S. Barik), gs13@iitbbs.ac.in (G. Sahoo).

https://doi.org/10.1016/j.akcej.2017.11.004
0972-8600/

Please cite this article in press as: S. Barik, G. Sahoo, Results on Laplacian spectra of graphs with pockets, AKCE International Journal of Graphs and Combinatorics (2017), https://doi.org/10.1016/j.akcej.2017.11.004.

From literature we find many operations defined on graphs such as disjoint union, complement, join, graph products (Cartesian product, direct product, strong product, lexicographic product, etc.), corona and many variants of corona (like edge corona, neighborhood corona, edge neighborhood corona, etc.). For such operations often it is possible to describe the Laplacian spectrum of the resulting graph using the Laplacian spectra of the corresponding constituting graphs, see [4,5] for reference. This enables one to visualize a complex network in terms of small simple recognizable graphs whose Laplacian spectra is easily computable. It is always interesting for researchers in spectral graph theory to define some new graph operations such that the Laplacian spectra of the new graphs produced can be described using the Laplacian spectra of the constituent graphs.

If $F = (V(F), E(F))$ and $H = (V(H), E(H))$ are two graphs on disjoint sets of m and n vertices, respectively, their union is the graph $F + H = (V(F) \cup V(H), E(F) \cup E(H))$, and their join is $F \vee H = (F^c + H^c)^c$, the graph on $m + n$ vertices obtained from $F + H$ by adding new edges from each vertex of F to every vertex of H. Note that, here F^c denotes the complement graph of the graph F.

The following result which describes the Laplacian spectrum of join of two graphs is often used in the next sections.

Theorem 1 (*[6], Theorem 2.1*)**.** *Let F, H be graphs on disjoint sets of m, n, vertices respectively, and $G = F \vee H$. Let $\sigma(F) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\sigma(H) = (\mu_1, \mu_2, \ldots, \mu_n)$. Then $0, n+\lambda_2, \ldots, n+\lambda_m, m+\mu_2, \ldots, m+\mu_n, m+n \in \sigma(G)$.*

The following graph operation is defined in [7].

Definition 2 (*[7]*)**.** Let F, H_v be connected graphs, v be a specified vertex of H_v and $u_1, \ldots, u_k \in F$. Let $G = G[F, u_1, \ldots, u_k, H_v]$ be the graph obtained by taking one copy of F and k copies of H_v, and then attaching the ith copy of H_v to the vertex u_i, $i = 1, \ldots, k$, at the vertex v of H_v (identify u_i with the vertex v of the ith copy). Then the copies of the graph H_v that are attached to the vertices u_i, $i = 1, \ldots, k$ are referred to as pockets, and G is described as a graph with pockets.

Suppose that F and H_v are graphs on n and $m + 1$ vertices, respectively. Using the above operation, we can produce many interesting classes of graphs. Since the operation is a very general operation, in [7], the author has asked the question that 'how far can the Laplacian spectrum of G be described by using the Laplacian spectra of F and H_v?'. In that paper, the author has described the Laplacian spectrum of G using the Laplacian spectra F and H_v in a particular case when $\deg(v) = m$. This motivates us for studying the Laplacian spectrum of more such graphs relaxing condition that $\deg(v) = m$. Let $\deg(v) = l$, $1 \le l \le m$. In this case, we denote $G[F, u_1, \ldots, u_k, H_v]$ more precisely by $G[F, u_1, \ldots, u_k; H_v, l]$. When $k = n$, we denote $G = G[F, u_1, \ldots, u_k; H_v, l]$ simply by $G[F; H_v, l]$.

Let $H = H_v \setminus \{v\}$. When $\deg(v) = m$, we have $H_v = \{v\} \vee H$. Now using Theorem 1, the Laplacian spectrum of H_v can be obtained from that of H. So the question of describing the Laplacian spectrum of G in terms of the Laplacian spectra of F and H_v is same as asking for the description of the Laplacian spectrum of G in terms of the Laplacian spectra of F and H. Further, when $l = m$ and $k = n$, we have $G[F; H_v, m] = F \circ H$, the corona of F and H (see [8]). In [9], the Laplacian spectrum of $F \circ H$ has been completely described using the Laplacian spectra of F and H.

Now if $\deg(v) = l$, $1 \le l \le m$, let $N(v) = \{v_1, v_2, \ldots, v_l\} \subset V(H_v)$, be the neighborhood set of v in H_v. Let H_1 be the subgraph of H_v induced by the vertices in $N(v)$ and H_2 be the subgraph of H_v induced by the vertices which are in $V(H_v) \setminus (N(v) \cup \{v\})$. When $H_v = H_1 \vee (H_2 + \{v\})$, we describe the complete Laplacian spectrum of $G[F; H_v, l]$ using that of F and H. The results are contained in Section 2 of the article.

Further, except $n + 2k$ eigenvalues, we describe all other eigenvalues of $G[F, u_1, \ldots, u_k; H_v, l]$ and prove that the remaining $n + 2k$ eigenvalues are independent of the graphs H_1 and H_2. In particular, when $F = F_1 \vee F_2$, where F_1 is the subgraph of F induced by the vertices $u_1, \ldots, u_k$, we describe the complete Laplacian spectrum of $G[F, u_1, \ldots, u_k; H_v, l]$. These results are contained in Section 3.

Following notations are being used in rest of the paper. The $n \times 1$ vector with each entry 0 is denoted by $\mathbf{0}_n$. The *Kronecker product* of matrices $R = [r_{ij}]$ and S is defined to be the partitioned matrix $[r_{ij}S]$ and is denoted by $R \otimes S$. The vector with ith entry equal to 1 and all other entries zero is denoted by e_i. By $J_{n\times m}$ we denote the matrix of order $n \times m$ whose entries are all equal to 1. If $n = m$, we write $J_{n\times m}$ simply by J_n. By I_n we denote the identity matrix of size n. We avoid writing the order of these matrices if it is clear from the context. K_n and C_n denote the complete graph and cycle graph of order n, respectively.