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On the second minimum algebraic connectivity of the graphs whose complements are trees

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Abstract

For a graph Γ the algebraic connectivity denoted by $a(\Gamma)$, is the second smallest eigenvalue of the Laplacian matrix of Γ. In Jiang et al. (2015), proved a unique graph with first minimum algebraic connectivity among the graphs which belong to a class of graphs whose complements are trees. In this paper, we characterize the unique graph with second minimum algebraic connectivity in the same aforesaid class of graphs.

Keywords: Laplacian matrix; Algebraic connectivity; Complement of trees

1. Introduction

In this work, we are concerned with simple, finite and undirected graphs. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be such a graph with vertex set $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(\Gamma)$. The adjacency matrix $A(\Gamma)$ of Γ is a square matrix of order n whose (i, j)-entry is unity if the vertices v_i and v_j are adjacent, and is zero otherwise. The eigenvalues and eigenvectors of the adjacency matrix of a graph are called ordinary eigenvalues and eigenvectors and the theory related to these eigenvalues and eigenvectors is usually referred to *graph spectral theory* which is well developed nowadays, see [1,2].

The degree $d_\Gamma(v_i)$ of the vertex v_i is the number of the first neighbors of this vertex in Γ. By $D(\Gamma)$, we denote the square matrix of order n whose ith diagonal element is equal to d_i and whose off-diagonal elements are zero. The Laplacian matrix of the graph Γ is

$$L(\Gamma) = D(\Gamma) - A(\Gamma)\,. \tag{1}$$

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For $i \in \{1, 2, 3, \dots, n\}$, the eigenvalues denoted by $\mu_i = \mu_i(\Gamma)$ and eigenvectors denoted by $X_i = X_i(\Gamma)$ of the Laplacian matrix $L(\Gamma)$ are the Laplacian eigenvalues and the Laplacian eigenvectors of the graph Γ. Suppose that the n-dimensional column-vectors $X_i \neq 0$ for $i = 1, 2, \dots, n$, then we have the following equality

$$L(\Gamma)X_i = \mu_i X_i.$$

Since $L(\Gamma)$ is a symmetric and real matrix, the Laplacian eigenvalues are non-negative real numbers. Thus we can arrange the Laplacian eigenvalues as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$, where μ_n is a minimum Laplacian eigenvalue which is always zero. The eigenvalue $\mu_{n-1}(\Gamma) = a(\Gamma)$ is called the *algebraic connectivity* of the graph Γ and it is a positive value if and only if Γ is connected. Thus, the algebraic connectivity $a(\Gamma)$ is a good parameter to measure how well a graph is connected and plays an important role in control theory and communications, and so forth. In particular, it is related to the synchronization ability of complex network [3,4]. On the other hand it is related to the convergence speed in networks; one important topic is to increase $a(\Gamma)$ as much as possible. There are several techniques to enhance this metric; see [5].

The eigenvectors corresponding to $a(\Gamma)$ are called *Fiedler* vectors of Γ. For graph Γ the complement of Γ is denoted by $\Gamma^c = (V(\Gamma), \bar{E}(\Gamma))$, where $\bar{E}(\Gamma) = \{xy : x, y \in V(\Gamma), xy \notin E(\Gamma)\}$. If Γ is a connected graph and Γ^c is its complement then every Laplacian eigenvector of Γ is a Laplacian eigenvector of Γ^c [6]. The matrix $Q(\Gamma) = D(\Gamma) + A(\Gamma)$ is called the signless Laplacian matrix of Γ. For further study, we refer [7,8].

A connected graph Γ with n vertices and m edges is called $k-cyclic$ graph if $m = n - 1 + k$. In particular if $k = 0$, then Γ is called a tree. Let $\mathcal{T}_n$ be the set of trees on n vertices. We denote by $K_{1,n-1}$ the star graph on n vertices. Define $\Im_n = \{\Gamma : \Gamma \text{ is a tree of order } n \text{ and } \Gamma \ncong K_{1,n-1}\}$ and $\Im_n^c = \{\Gamma^c : \Gamma \in \Im_n\}$.

When the structures of graphs become complex comparatively to their complements, then we study the graphs by studying their complements. Some following results are obtained in the spectral graph theory using this approach. Fan et al. [9] characterized the unique graph with minimum least eigenvalue in the class of graphs $\Im_n^c$. In the same class whose complements are trees, Li and Wang [10] proved the unique graph whose signless Laplacian least eigenvalue attains the minimum. Moreover, Yu and Fan [11,12] found two unique graphs whose least eigenvalue attains the minimum in the class of graphs whose complements are connected and have no cut edges or cut vertices.

Recently, Jiang, Yu and Cao [13] studied the unique graph $T(p, q)$ which has first minimum algebraic connectivity among all the graphs which belong to $\Im_n^c$, where $T(p, q)$ is obtained from two disjoint stars $K_{1,p}(p \geq 1)$ and $K_{1,q}(q \geq 1)$ by joining the center of $K_{1,p}$ to the center of $K_{1,q}$ with an edge. In this paper, for the same class, we characterize the unique graph which has second minimum algebraic connectivity. The rest of the paper is organized as follows: In Section 2, we present some basic definitions, notions and lemmas which are used in the main results and Section 3 includes the main results related to the second minimum algebraic connectivity of the graphs whose complements are trees.

2. Preliminaries

We begin with some definitions. For a graph Γ of order n, a vector $X \in R^n$ is called to be defined on Γ, if there exists a one–one map $\gamma : V(\Gamma) \to X$, simply written $X_u = \gamma(u)$ for each $u \in V(\Gamma)$. If X is an eigenvector of $L(\Gamma)$, then naturally X is defined on Γ, X_u denotes the entry of X corresponding to the vertex u. Such labelings are sometimes called valuations of the vertices of Γ. One can easily find that, for an arbitrary vector $X \in R^n$,

$$X^T L(\Gamma)X = \sum_{uv \in E(\Gamma)} (X_u - X_v)^2, \tag{2}$$

and when λ is a Laplacian eigenvalue of Γ corresponding to the eigenvector X then we have,

$$(d_\Gamma(v) - \lambda)X_v = \sum_{u \in N_\Gamma(v)} X_u, \text{ for each vertex } v \in V(\Gamma), \tag{3}$$

if and only if $X \neq 0$. In the above equation $N_G(v)$ denotes the neighbor vertices of $v \in \Gamma$. Eq. (3) is called the Laplacian eigenvalue-equation for the graph Γ.

Further by the well known Courant–Fisher theorem [14], for an arbitrary unit vector $X \in R^n$, when $X \neq 0$ and $X \perp 1$, then

$$a(\Gamma) \leq X^T L(\Gamma)X, \tag{4}$$

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