Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics 15 (2018) 105–111
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

On edge-graceful labeling and deficiency for regular graphs

Tao-Ming Wang[a,*], Guang-Hui Zhang[b]

[a] Department of Applied Mathematics, Tunghai University, Taichung 407, Taiwan
[b] Department of Applied Mathematics, National Chung Hsing University, Taichung 402, Taiwan

Available online 24 April 2018

Abstract

An edge-graceful labeling of a finite simple graph with p vertices and q edges is a bijection from the set of edges to the set of integers $\{1, 2, \dots, q\}$ such that the vertex sums are pairwise distinct modulo p, where the vertex sum at a vertex is the sum of labels of all edges incident to such vertex. A graph is called edge-graceful if it admits an edge-graceful labeling. In 2005 Hefetz (2005) proved that a regular graph of even degree is edge-graceful if it contains a 2-factor consisting of mC_n, where m, n are odd. In this article, we show that a regular graph of odd degree is edge-graceful if it contains either of two particular 3-factors, namely, a claw factor and a quasi-prism factor. We also introduce a new notion called edge-graceful deficiency, which is a parameter to measure how close a graph is away from being an edge-graceful graph. In particular the edge-graceful deficiency of a regular graph of even degree containing a Hamiltonian cycle is completely determined.

Keywords: Edge-graceful; Edge-graceful deficiency; Claw factor; Quasi-prism; Hamiltonian cycle

1. Introduction and background

All graphs in this paper are finite simple, undirected, and without loops unless otherwise stated. In 1990, N. Hartsfield and G. Ringel [1] introduced the concepts called antimagic labeling and antimagic graphs.

Definition 1. Let $G = (V, E)$ be a graph with p vertices, q edges, and without any isolated vertex. An **antimagic** edge labeling is a bijection $f : E \to \{1, 2, \dots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{Z}^+$ given by $f^+(u) = \sum\{f(uv) : uv \in E\}$ is injective. A graph is called **antimagic** if it admits an antimagic labeling. If moreover for G the vertex sums f^+ are consecutive integers, we say G admits an $(a, 1)$**-antimagic** labeling and G is an $(a, 1)$**-antimagic** graph.

Definition 2. Let $G = (V, E)$ be a graph with p vertices, q edges, and without any isolated vertex. An **edge-graceful** edge labeling is a bijection $f : E \to \{1, 2, \dots, q\}$, such that the induced vertex sum $f^+ : V \to \mathbb{Z}_p$ given by

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail address: wang@go.thu.edu.tw (T.-M. Wang).

https://doi.org/10.1016/j.akcej.2018.03.002

$f^+(u) = \sum\{f(uv) : uv \in E\}\ (mod\ p)$ is injective. A graph is called **edge-graceful** if it admits an edge-graceful labeling.

Note that an $(a, 1)$-antimagic labeling is an edge-graceful labeling, and an edge-graceful labeling is necessarily an antimagic labeling.

In 1985 S.P. Lo [2] introduced such notion of edge-graceful labeling. In 2005 D. Hefetz [3] proved that, for an edge-graceful graph G, it is still edge-graceful after adding an arbitrary even factor. In 2008, T.-M. Wang [4] studied edge-graceful spectrum of graphs. Most recently M. Bača et al. [5] studied the existence for $(a, 1)$-antimagic-ness of certain 3-regular graphs. Many various types of graphs have been shown to be antimagic [3–15] over the years. For more conjectures and open problems on various types of antimagic labeling problems, interested readers can refer to the dynamic survey article of J. Gallian [16].

In this paper, we show that an odd regular graph is edge-graceful if it contains a quasi-prism factor or a claw factor. Also a more general concept called edge-graceful deficiency is introduced, and we completely determine the edge-graceful deficiency of Hamiltonian regular graphs of even degree.

2. Odd regular graphs with a quasi-prism factor

The following is a necessary condition for a graph G to be an edge-graceful graph:

Lemma 3 ([2]). *Let G be a graph with p vertices and q edges. If G is edge-graceful, then $q(q+1) \equiv \frac{p(p-1)}{2}\ (mod\ p)$.*

In 2005 D. Hefetz [3] proved the following:

Lemma 4 (D. Hefetz [3]). *Assume H is a graph which is obtained from a graph G by adding an arbitrary even factor. If G is edge-graceful, then H is edge-graceful.*

According to the 2-factorization of an even regular graph by Petersen [17], the edge-graceful-ness of regular graphs can be reduced to the study of that of 2-regular graphs and 3-regular graphs. D. Hefetz mentioned the following sufficient condition for edge-graceful even regular graphs:

Corollary 5 (D. Hefetz [3]). *Assume H is an even regular graph which contains a 2-factor consisting of mC_n, where m, n are odd. Then H is edge-graceful.*

In this section we show that an odd regular graph is edge-graceful if it contains a quasi-prism factor, which is a particular 3-regular spanning subgraph. It is clear that a 3-regular graph has even number of vertices. We define quasi-prisms as follows:

Definition 6. A **quasi-prism** $G(U, V)$ is a 3-regular graph with $2k$ vertices $U = \{u_1, u_2, \ldots, u_k\}$ and $V = \{v_1, v_2, \ldots, v_k\}$, which consists of the edge-disjoint union of a 1-factor and two 2-regular subgraphs induced by U and V respectively. (**See** Fig. 1.)

The generalized Petersen graphs $GP(n, k)$ are examples of quasi-prisms $G(U, V)$ for which 2-regular subgraphs induced by U and V are defined as follows.

Definition 7. Let n, k be integers such that $n \geq 3$ and $1 \leq k \leq \lfloor\frac{n-1}{2}\rfloor$. The generalized Petersen graph $GP(n, k)$ is defined by $V(GP(n, k)) = \{u_i, v_i \mid 1 \leq i \leq n\}$, and $E(GP(n, k)) = \{u_iu_{i+1}, u_iv_i, v_iv_{i+k} \mid 1 \leq i \leq n\}$ where the subscripts are taken modulo n. (**See** Fig. 2.) We call $u_1, u_2, \ldots, u_n$ an outer cycle, and $v_1, v_2, \ldots, v_n$ an inner cycle.

Now, we are in a position to show the following:

Theorem 8. *All quasi-prisms are edge-graceful.*

Proof. We have a convention that the notation $[a, b]$ stands for the set of all integers between a and b (inclusive), namely, $\{a, a + 1, \ldots, b\}$. Let $G(U, V)$ be a quasi-prism with $4n$ vertices and $6n$ edges. Also let $G(U, V)$ be the 3-regular graph with $4n$ vertices $U = \{u_1, u_2, \ldots, u_{2n}\}$ and $V = \{v_1, v_2, \ldots, v_{2n}\}$, which consists of the edge-disjoint union of a 1-factor and two 2-regular subgraphs induced by U and V respectively. We denote by $G[U]$ and