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Some new upper bounds of $ex(n; \{C_3, C_4\})$

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Abstract

The extremal number $ex(n; \{C_3, C_4\})$ or simply $ex(n; 4)$ denotes the maximal number of edges in a graph on n vertices with forbidden subgraphs C_3 and C_4. The exact number of $ex(n; 4)$ is only known for n up to 32 and $n = 50$. There are upper and lower bounds of $ex(n; 4)$ for other values of n. In this paper, we improve the upper bound of $ex(n; 4)$ for $n = 33, 34, \ldots, 42$ and also $n = d^2 + 1$ for any positive integer $d \neq 7, 57$.

Keywords: Extremal graph; Extremal number; Forbidden subgraph

1. Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order and the size of a graph are defined to be the number of vertices and edges, respectively. The number of edges incident to a vertex v is called the degree of v ($d(v)$). If all vertices in G have the same degree r, then G is said to be *regular* or more specifically, r-regular. Let $\delta(\Delta)$ denote the minimum (maximum) degree of a graph and girth, $g(G)$, denote the length of the smallest cycle in a graph.

The *distance* between 2 vertices in a graph is defined to be the smallest number of edges connecting those two vertices. The maximum distance from a vertex to all other vertices is called the *eccentricity* of a vertex. The *diameter*, D, of a graph G is the maximum eccentricity over all the vertices in a graph.

In this paper, we discuss the size maximality of graphs with some constraints. The graphs are required to have maximum number of edges without containing some given subgraphs. In general, let $\mathcal{F}$ be a family of graphs. A graph is called $\mathcal{F}$-free if it does not contain any subgraph isomorphic to any of the graphs in $\mathcal{F}$. The typical question that arises is:

"*How many edges can an $\mathcal{F}$-free graph with n vertices have?*"

The maximum number of edges in an $\mathcal{F}$-free graph on n vertices is denoted by $ex(n; \mathcal{F})$. The family of graphs $\mathcal{F}$ itself is called "family of forbidden subgraphs". There are several variations of $\mathcal{F}$ that have been considered, including

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triangles, cliques, squares, claw graphs ($K_{1,3}$), and the set of cycles of given lengths. The first forbidden subgraph considered was a triangle, $\mathcal{F} = \{C_3\}$.

Initiated by Mantel's Theorem in 1907, if a graph G on n vertices contains more than $n^2/4$ edges, then G contains a triangle. Hence, the extremal number of triangle-free graphs is at most $\lfloor n^2/4 \rfloor$. This bound is obtained by the complete bipartite graphs $K_{\lfloor n/2 \rfloor \lceil n/2 \rceil}$.

In 1975, Erdős posed the problem of finding the maximum number of edges in graphs with n vertices that do not contain C_3 or C_4. By $ex(n; \{C_3, C_4\})$, or simply $ex(n; 4)$, we mean the maximum number of edges in a graph of order n and girth $g \geq 5$. A graph that has $ex(n; 4)$ edges is called an *extremal graph*. The set of all those extremal graphs is denoted by $EX(n; 4)$.

Erdős in [1] conjectured that $ex(n; 4) = (1/2 + o(1))^{3/2} n^{3/2}$. In [2], Wang constructed regular graphs of degree $d = 2^k + 1$ and $n = 2d^2 - 4d + 2$. The number of edges in these graphs also attains the best-known lower bound and asymptotically approaches the value in Erdős' conjecture. Garnick et al. in [3] gave the exact value of $ex(n; 4)$ for all n up to 24 and constructive lower bound for all n up to 200 by employing an algorithm involving both hill-climbing and backtracking techniques. They also enumerated all of the extremal graphs of order less than 21. Most of their results were verified by McKay's Nauty program [4,5]. Additional values of $ex(n; 4)$ for $25 \leq n \leq 30$ were determined by Garnick and Nieuwejaar [6]. The upper bound for this problem is the following [7]: $ex(n; \{C_3, C_4\}) \leq \frac{n\sqrt{n-1}}{2}$.

In Section 2, we mention some properties of extremal graphs of girth 5 that will be useful in our proofs. In Section 3, we discuss the new upper bound of $ex(n; 4)$, for $n = 33, \ldots, 42$ and $n = d^2 + 1$ for any positive integer $d \neq 7, 57$. In the end of this paper, we summarise the known exact values, the lower and the upper bound for the extremal number in Table 1.

2. Properties of extremal graphs

The girth of a graph is related to its diameter by $g \leq 2D + 1$. Since we are dealing with graphs of girth at least 5, then for $n \geq 3$, the diameter is at least 2. The upper bound of the diameter of the extremal graphs is given in Proposition 2.1.

Proposition 2.1 ([3]). *Let G be an extremal $\{C_3, C_4\}$-free graph of order n.*

1. *The diameter of G is at most 3.*
2. *Suppose that the minimum degree of graph G is equal to 1 and let x be a vertex with degree 1, $d(x) = \delta(G) = 1$, then the graph $G - \{x\}$ has diameter at most 2.*

Some parameters of an extremal graph are related to its extremal number as stated in the following proposition.

Proposition 2.2 ([3]). *For all $\{C_3, C_4\}$-free graphs G of order $n \geq 1$ and m edges, then*

1. $n \geq 1 + \Delta\delta \geq \delta^2$;
2. $\delta \geq m - ex(n-1; 4)$ *and* $\Delta \geq \lceil \frac{2m}{n} \rceil$;
3. $n \geq 1 + \lceil 2ex(n; 4)/n \rceil (ex(n; 4) - ex(n-1; 4))$.

Proposition 2.2 shows that the knowledge of the extremal number of order $n - 1$ is indeed useful in determining the extremal number of order n. For example, it gives a bound for the degree. The second point in Proposition 2.2 says that if we have a graph with minimum degree δ and size m, then $\delta \geq m - ex(n-1; 4)$. This guarantees that removing a vertex with degree δ will not give a graph of size more than $ex(n-1; 4)$. Besides considering removing a single vertex from a graph, Garnick and Nieuwejaar considered the removal of a larger subgraph from the graph, as in Proposition 2.3.

Proposition 2.3 ([6]). *For any k vertices $x_1, x_2, \ldots, x_k$ in a $\{C_3, C_4\}$-free graph G with order $n > 1$ and size $m > 1$, let $d(x_i)$ be the degree of vertex x_i and $\langle x_1, x_2, \ldots, x_k \rangle$ be the vertex induced subgraph of G, then*

$$\sum_{i=1}^{k} d(x_i) - |E(\langle x_1, x_2, \ldots, x_k \rangle)| \geq m - ex(n-k; 4)$$

where $|E(\langle x_1, x_2, \ldots, x_k \rangle)|$ denotes the number of edges in the induced subgraph.