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On total directed graphs of non-commutative rings

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Abstract

For a non-commutative ring R, the left total directed graph of R is a directed graph with vertex set as R and for the vertices x and y, x is adjacent to y if and only if there is a non-zero $r \in R$ which is different from x and y, such that $rx + yr$ is a left zero-divisor of R. In this paper, we discuss some very basic results of left (as well as right) total directed graph of R. We also study the coloring of left total directed graph of R directed graph.

Keywords: Non-commutative ring; Zero-divisor; Directed graph; Total graph; Clique

1. Introduction

The total graph of a commutative ring with unity was introduced by Anderson and Badawi in [1]. They considered the total graph $T(\Gamma(R))$ of a commutative ring R as an undirected graph with vertex set as R and any two vertices of $T(\Gamma(R))$ are adjacent if and only if their ring sum is a zero-divisor of R. They studied the characteristics of total graph and its two induced subgraphs by taking the cases when the set of zero-divisors of R is an ideal of R and when this set is not an ideal of R. In [2], Akbari et al. continued this concept of total graph. Ahmad Abbasi and Shokoofe Habibi [3] investigated the total graph of a commutative ring with respect to proper ideals. Anderson and Badawi [4] studied the total graph of a commutative ring without zero element. M. H. Shekarriz et al. observed some basic graph theoretic properties of the total graph of a finite commutative ring in [5]. The insight for total graph is extended to modules also. The total graph of a commutative ring with respect to the proper submodules of a module was interpreted by A. Abbasi and S. Habibi in [6]. The total torsion element graph of a module over a commutative ring was discussed by S. Atani and S. Habibi in [7]. These kinds of continuous extension of Anderson and Badawi's [1] work signify its utility in graphical aspects of ring-theoretic structures.

To interpret this concept in directed graph, we take the definition in a different way. We introduce left and right total directed graphs of non-commutative rings. The left total directed graph of a non-commutative ring R, denoted

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by $T_l(\Gamma(R))$, is a directed graph with all elements of R as vertices. Any $x, y \in R$, the vertex x is adjacent to y if and only if there exists a non-zero r, which is not equal to x and y, in R such that $rx + yr$ is a left zero-divisor of R. More specifically, the non-zero element r is considered either a (left) zero-divisor or a (left) regular element. In this paper, we take the non-zero element r as left non-zero zero-divisor for the adjacency relation of two vertices. Henceforth, we continue this discussion by taking the non-zero element r as left non-zero zero-divisor. The induced subgraphs $Z_l(\Gamma(R))$ and $Reg_l(\Gamma(R))$ of $T_l(\Gamma(R))$ with vertex sets $Z_l(R)$ (set of left zero-divisors of R) and $Reg_l(R)$ (set of left regular elements of R) respectively, are studied. The idea of its dual concept that is, right total directed graph $T_r(\Gamma(R))$ and its respective induced subgraphs $Z_r(\Gamma(R))$ and $Reg_r(\Gamma(R))$ of a non-commutative ring R, are also taken under consideration. We do not state or establish each and every dual concept in our discussion, as almost all results hold for the respective dual concept.

Note that the non-zero zero-divisor r is not independent for the choice of adjacency of two vertices. That is the underlying (undirected) graph of the directed graph is not simple. Recollect a simple (undirected) graph contains at most one edge (arc is called an edge in an undirected graph) between a pair of vertices. Though a part of our discussions is on basic characteristics of ring-theoretic concepts, yet this motivates us for the establishment of interesting results.

2. Preliminaries

Throughout our discussion, unless otherwise specified, rings mean non-commutative rings. From this onward, R denotes a non-commutative ring, G is a directed graph, and for any $A \subseteq R$, A^* contains all non-zero elements of A.

A digraph or directed graph G is a non-empty set of vertices, denoted by $V(G)$, and a collection of ordered pairs of distinct vertices. Any such pair (u, v) is called an arc or directed line and will usually be denoted uv or u adj v. If uv and vu are not arcs in G, then we say that u and v are not adjacent. If $a = uv$ is an arc of G, a is said to be incident out of u and incident into v. The number of arcs incident out of a vertex v is the out-degree of v and the number of arcs incident into v is its in-degree. A vertex v of G is called a sink, if the in-degree of v is positive and the out-degree of v is zero. The dual concept of sink is called source. G is said to be symmetric whenever u adj v, v adj u for the vertices u and v of G.

A walk $v_0x_1v_1 \cdots x_nv_n$, in a directed graph is an alternating sequence of vertices and arcs, in which each arc x_i is $v_{i-1}v_i$. The length of such a walk is n, the number of occurrences of arcs in it. A closed walk has the same first and last vertices. A path is a walk in which all vertices are distinct; a cycle or circuit is a closed walk with all vertices distinct (except the first and last). If there is a path from a vertex u to a vertex v, then v is said to be reachable from u. A digraph is strongly connected, if every two vertices are mutually reachable. G is said to be totally disconnected, if no two vertices of G are adjacent. For vertices x and y of G, we define $d(x, y)$ to be the length of any shortest path from x to y. The diameter $diam(G)$ of a graph G is defined as $max\{d(u, v) : u, v \in V(G)\}$. We say that two (induced) subgraphs G_1 and G_2 of G are disjoint if G_1 and G_2 have no common vertices and no vertex of G_1 (respectively, G_2) is adjacent (in G) to any vertex not in G_1 (respectively, G_2). A digraph is a tournament if its underlying graph is a complete graph i.e. any two vertices of the underlying graph are adjacent. It must be noted that this underlying graph is simple. Since the non-zero element r is not independent in adjacency of two vertices in total directed graph, so an underlying graph of the respective directed subgraph may not be simple. Thus for the results of tournament of this paper, we consider those rings whose underlying graph of corresponding (left) total directed graph is simple.

The coloring of a directed graph is an assignment of colors to the vertices of the graph with no two adjacent vertices are of same color, i.e. if $u\ adj\ v$ for the vertices u and v of a directed graph G, then u and v are of different colors. A clique of a directed graph is a maximal tournament subgraph. The minimum number of colors needed for proper coloring of a directed graph G is known as chromatic number of G, and is denoted by $\chi(G)$.

Next, we remember some definitions from ring-theoretic concepts. An element e_l of R is called left identity if $e_lx = x$, for all $x \in R$. The collection of all left identity elements of R is denoted by $E_l(R)$. The dual concept of left identity element is right identity element, and the set of all right identity elements of R is denoted by $E_r(R)$. An identity element of R is an element which is both a left identity element and a right identity element. If a ring has an identity element (unity), then it is unique. An element x of R is said to be a left inverse of an element y if $xy = e$, where e is unity of R. In the same way, right inverse can be defined. An invertible element of R is an element which has a left and right inverse such that both are equal. An element $a \in R$ is said to be left zero-divisor if there exists a non-zero element $b \in R$ such that $ab = 0$ in R. Right zero-divisors are defined similarly. The collection of left and right zero-divisors of R are denoted by $Z_l(R)$ and $Z_r(R)$ respectively. The elements, which are not left and right

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