Available online at www.sciencedirect.com
ScienceDirect
AKCE International Journal of Graphs and Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮
AKCE International Journal of Graphs and Combinatorics
www.elsevier.com/locate/akcej

m-dominating k-ended trees of l-connected graphs

Masao Tsugaki[a,*], Guiying Yan[b]

[a] Tokyo University of Science, Kagurazaka, Shinjuku-ku, Tokyo, Japan
[b] Institute of mathematical and system sciences, Chinese Academy of Science, Beijing, PR China

Received 29 August 2015; accepted 19 March 2017
Available online xxxxx

Abstract

Let $k \geq 2$, $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If there exists a subgraph X of G such that the distance between v and X is at most m for any $v \in V(G)$, then we say that X m-dominates G. A subset S of $V(G)$ is said to be $2(m+1)$-stable if the distance between each pair of distinct vertices in S is at least $2(m+1)$. In this paper, we prove that if G does not have a $2(m+1)$-stable set of order at least $k+l$, then G has an m-dominating tree which has at most k leaves.

Keywords: Vertex dominating; m-dominating; k-ended tree

1. Introduction

In this paper, we consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph, and X be a subgraph or a vertex set of G. We write $|X|$ for the order of X. For two vertices u and v of G, let $d_G(u, v)$ denote the distance between u and v. For a vertex v of G, the distance between v and X is defined to be the minimum value of $d_G(v, x)$ for all $x \in V(X)$ or $x \in X$, and denoted by $d_G(v, X)$. For an integer $m \geq 0$, let $Domi^m(X) := \{v \in V(G) : d_G(v, X) \leq m\}$. If $V(G) = Domi^m(X)$, then we say that X *m-dominates* G. For an integer $l \geq 2$, a subset S of $V(G)$ is said to be *l-stable* if the distance between each pair of distinct vertices in S is at least l. The l-stable number of G is the cardinality of a maximum l-stable set of G, and is denoted by $\alpha^l(G)$, that is $\alpha^l(G) := \max\{|S| : S \subseteq V(G), S \text{ is } l\text{-stable}\}$. Note that $\alpha^2(G)$ is the independence number of G. A tree is called a *k-ended tree* if the number of leaves is at most k. In this paper, we investigate a stable number condition for l-connected graphs to have m-dominating k-ended trees. Concerning on this, the following three theorems are known.

Theorem 1 (*Kano, Tsugaki and Yan [1]*). *Let $k \geq 2$ and $m \geq 0$ be integers, and let G be a connected graph. If $\alpha^{2(m+1)}(G) \leq k$, then G has an m-dominating k-ended tree.*

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: tsugaki@hotmail.com (M. Tsugaki), yangy@amss.ac.cn (G. Yan).

http://dx.doi.org/10.1016/j.akcej.2017.04.004
0972-8600/

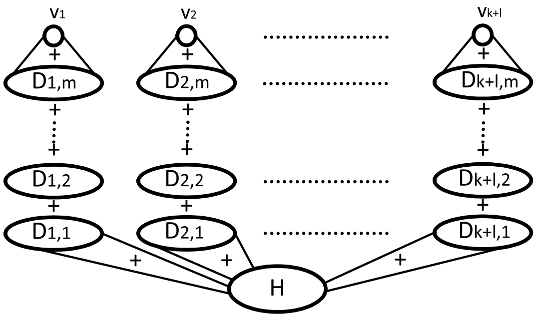

Fig. 1. An l-connected graph which has no m-dominating k-ended tree.

Theorem 2 (*Broersma [2]*)**.** *Let $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If $\alpha^{2(m+1)}(G) \leq l+1$, then G has an m-dominating path, that is an m-dominating 2-ended tree.*

Theorem 3 (*Win [3]*)**.** *Let $k \geq 2$ and $l \geq 1$ be integers, and let G be an l-connected graph. If $\alpha^2(G) \leq k+l-1$, then G has a spanning k-ended tree, that is a 0-dominating k-ended tree.*

In this paper, we prove the following theorem which is a generalization of above three theorems.

Theorem 4. *Let $k \geq 2$, $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If $\alpha^{2(m+1)}(G) \leq k+l-1$, then G has an m-dominating k-ended tree.*

The upper bound of the stable number condition of Theorem 4 is sharp. Let $k \geq 2$, $l \geq 1$ and $m \geq 1$ be integers. For $1 \leq i \leq k+l$, let $D_{i,1}, D_{i,2}, \ldots, D_{i,m}$ be disjoint copies of the complete graph K_l. For $1 \leq j \leq m-1$, join all the vertices of $D_{i,j}$ and all the vertices of $D_{i,j+1}$ by edges. Let v_i be a vertex not contained in $D_{i,1} \cup D_{i,2} \cup \cdots \cup D_{i,m}$, and join v_i and all the vertices of $D_{i,m}$ by edges. Let H be a complete graph of order l. For $1 \leq i \leq k+l$, join all the vertices of H and all the vertices of $D_{i,1}$ by edges. Let G denote the resulting graph (see Fig. 1). Then G is an l-connected graph, and has no m-dominating k-ended tree. On the other hand, $\alpha^{2(m+1)}(G) = k+l$.

Concerning on a degree sum condition, the following theorem is known.

Theorem 5 (*Broersma [2]*)**.** *Let $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If $\sum_{x \in S} |Domi^m(x)| \geq |G| - l + 1$ for any $2(m+1)$-stable set S of order $l+2$, then G has an m-dominating path, that is an m-dominating 2-ended tree.*

By Theorem 4, we can obtain the following theorem which is a generalization of Theorem 5.

Theorem 6. *Let $k \geq 2$, $l \geq 1$ and $m \geq 0$ be integers, and let G be an l-connected graph. If $\sum_{x \in S} |Domi^m(x)| \geq |G| - l + 1$ for any $2(m+1)$-stable set S of order $k+l$, then G has an m-dominating k-ended tree.*

Proof of Theorem 6. Suppose that G does not have an m-dominating k-ended tree. Then, by Theorem 4, $\alpha^{2(m+1)}(G) \geq k+l$, and hence there exists a $2(m+1)$-stable set S of order $|S| = k+l$. Note that for any $x, y \in S$ with $x \neq y$, $Domi^m(x)$ and $Domi^m(y)$ are disjoint and there are no edge between them. Since G is l-connected, these imply that $|G| - \sum_{x \in S} |Domi^m(x)| \geq l$, which contradicts to the condition in Theorem 6. $\square$

2. Proof of Theorem 4

A *system* of a graph G is defined to be a set of vertex-disjoint paths and cycles of G. In this paper, we regard a vertex and an edge as a cycle, and thus a path means a path of order at least 3. Let $\mathcal{S}$ be a system of a graph G. For $S \in \mathcal{S}$, we put $f(S) := 2$ if S is a path, and $f(S) := 1$ otherwise. Let $\mathcal{P}_{\mathcal{S}} := \{S \in \mathcal{S} : f(S) = 2\}$ and $\mathcal{C}_{\mathcal{S}} := \{S \in \mathcal{S} : f(S) = 1\}$. For $\mathcal{X} \subseteq \mathcal{S}$ and an integer $m \geq 0$, we define $V(\mathcal{X}) := \bigcup_{X \in \mathcal{X}} V(X)$, $Domi^m(\mathcal{X}) := \bigcup_{X \in \mathcal{X}} Domi^m(X)$ and $f(\mathcal{X}) := \sum_{X \in \mathcal{X}} f(X)$. For an integer $k \geq 2$, we call $\mathcal{X}$ a *k-ended system* if $f(\mathcal{X}) \leq k$. Note that $f(\mathcal{S}) = 2|\mathcal{P}_{\mathcal{S}}| + |\mathcal{C}_{\mathcal{S}}|$.