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An improved bound for negative binomial approximation with z-functions

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Abstract

In this article, we use Stein's method together with z-functions to give an improved bound for the total variation distance between the distribution of a non-negative integer-valued random variable X and the negative binomial distribution with parameters $r \in \mathbb{R}^+$ and $p = 1 - q \in (0, 1)$, where $\frac{rq}{p}$ is equal to the mean of X, $E(X)$. The improved bound is sharper than that mentioned in Teerapabolarn and Boondirek (2010). We give three examples of the negative binomial approximation to the distribution of X concerning the negative hypergeometric, Pólya and negative Pólya distributions.

Keywords: Negative binomial distribution; Negative binomial approximation; Stein's method; z-function

1. Introduction and main result

Let X be a non-negative integer-valued random variable with mean $\mu = E(X)$ and variance $0 < \sigma^2 = Var(X) < \infty$ and have probability mass function $\wp_X(x) > 0$ for every x in the space of X, denoted by $\mathcal{X}$. Let $N_{r,p}$ be the negative binomial random variable with parameters $r \in \mathbb{R}^+$ and $p = 1 - q \in (0, 1)$ following the probability mass function

$$\wp_{N_{r,p}}(k) = \frac{\Gamma(r+k)}{\Gamma(r)k!} q^k p^r, \ k \in \mathbb{N} \cup \{0\}, \tag{1.1}$$

where $E(N_{r,p}) = \frac{rq}{p}$ and $Var(N_{r,p}) = \frac{rq}{p^2}$ are its mean and variance.

The research topic related to the context of negative binomial approximation was first proposed by Brown and Phillips [1]. They used Stein's method to give a bound on negative binomial approximation to the distribution of a sum of dependent Bernoulli random variables, and also applied the result to approximate the Pólya distribution. Vellaisamy and Upadhye [2] used Kerstan's method to give a bound on the negative binomial approximation to the

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distribution of a sum of independent geometric random variables. However, both the results are inappropriate for applying to a single non-negative integer-valued random variable X. Later, Teerapabolarn and Boondirek [3] used Stein's method together with z-functions to give a bound for approximating the distribution of a single non-negative integer-valued random variable X by a negative binomial distribution as follows:

$$d(X, N_{r,p}) \leq \frac{p(1-p^r)}{rq} E\left|\frac{(r+X)q}{p} - z(X)\right| + \left|\frac{rq}{p} - \mu\right| \tag{1.2}$$

and for $\frac{rq}{p} = \mu$,

$$d(X, N_{r,p}) \leq \frac{p(1-p^r)}{rq} E\left|\frac{(r+X)q}{p} - z(X)\right|, \tag{1.3}$$

where $d(X, N_{r,p}) = \sup_{A \subseteq \mathbb{N}\cup\{0\}} |P(X \in A) - P(N_{r,p} \in A)|$ is the total variation distance between the distributions of X and $N_{r,p}$ and z is z-function associated with the random variable X, $z(x) = \frac{1}{\wp_X(x)} \sum_{k=0}^{x} (\mu - k)\wp_X(k)$ for $x \in \mathcal{X}$.

In this study, we aim to improve the bound in (1.3) to be sharper. The following theorem is the main result.

Theorem 1.1. *Let $\frac{rq}{p} = \mu$, then the following inequality holds:*

$$d(X, N_{r,p}) \leq \min\left\{\frac{1-p^{r+1}}{(r+1)q} E\left|(r+X)q - z(X)p\right|, \sum_{x\in\mathcal{X}\setminus\{0\}} \left|1 - \frac{z(x)p}{(r+x)q}\right| \wp_X(x)\right\}. \tag{1.4}$$

The following corollary is a consequence of Theorem 1.1.

Corollary 1.1. *If $(r+x)q - z(x)p \geq 0$ for every $x \in \mathcal{X}$ or $(r+x)q - z(x)p \leq 0$ for every $x \in \mathcal{X}$, then we have the following:*

$$d(X, N_{r,p}) \leq \min\left\{\frac{1-p^{r+1}}{(r+1)q} \left|\mu - \sigma^2 p\right|, \sum_{x\in\mathcal{X}\setminus\{0\}} \left|1 - \frac{z(x)p}{(r+x)q}\right| \wp_X(x)\right\}. \tag{1.5}$$

Consider the bounds in (1.3) and (1.4), it can be seen that the bound in Theorem 1.1 is sharper than that presented in (1.3), because $\frac{1-p^{r+1}}{(r+1)q} < \frac{1-p^r}{rq}$ [4]. In addition, we also give three examples to illustrate applications of the result in approximating the negative hypergeometric, Pólya and negative Pólya distributions.

2. Proof of main result

Stein's method and z-functions are the tools for giving the main result, which are mentioned as follows.

For z-functions, following Teerapabolarn and Boondirek [3], a function z associated with the random variable X is of the form

$$z(x)\wp_X(x) = \sum_{i=0}^{x} (\mu - i)\wp_X(i), \;\; x \in \mathcal{X} \tag{2.1}$$

and a simple form of (2.1) is

$$z(0) = \mu, \;\; z(x) = \frac{z(x-1)\wp_X(x-1)}{\wp_X(x)} + \mu - x, \;\; x \in \mathcal{X} \setminus \{0\}. \tag{2.2}$$

For Stein's method of the negative binomial distribution, Brown and Phillips [1] introduced Stein's method to the negative binomial approximation. Stein's equation for the negative binomial distribution with parameters $r \in \mathbb{R}^+$ and $p = (1-q) \in (0,1)$ is of the form

$$h(x) - \mathbf{NB}_{r,p}(h) = q(r+x)f(x+1) - xf(x), \tag{2.3}$$

where $\mathbf{NB}_{r,p}(h) = \sum_{k=0}^{\infty} h(k) \frac{\Gamma(r+k)}{\Gamma(r)k!} p^r q^k$ and f and h are bounded real-valued functions defined on $\mathbb{N}\cup\{0\}$.

For $A \subseteq \mathbb{N}\cup\{0\}$, let $h_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{2.4}$$