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AKCE International Journal of Graphs and Combinatorics
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# A graph theoretic analysis of leverage centrality

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Received 12 August 2015; received in revised form 5 April 2016; accepted 8 May 2016
Available online xxxxx

**Abstract**

In 2010, Joyce et al. defined the leverage centrality of vertices in a graph as a means to analyze functional connections within the human brain. In this metric a degree of a vertex is compared to the degrees of all it neighbors. We investigate this property from a mathematical perspective. We first outline some of the basic properties and then compute leverage centralities of vertices in different families of graphs. In particular, we show there is a surprising connection between the number of distinct leverage centralities in the Cartesian product of paths and the triangle numbers.


*Keywords:* Centrality; Degrees; Triangle numbers

## 1. Introduction

In a social network people influence each other and those with lots of friends often have more leverage (or influence) than those with fewer friends. However the true influence of a person not only depends on the number of friends that they have, but also on the number of friends that their friends have. A person that is well connected can pass information to many friends, but if their friends are also receiving information from others, their influence on others is lessened. The extreme cases of influence occurs with a person who has a large number of friends, and for each of the friends, their only source of information is the original person. In this situation, the original person has the highest possible influence and all of the others have the lowest possible influence.

Peer review under responsibility of Kalasalingam University.
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http://dx.doi.org/10.1016/j.akcej.2017.05.001

The level of influence can be quantified by a property defined by Joyce et al. [1] known as *leverage centrality*. We recall that the degree of a vertex $v$ is the number of edges incident to $v$ and is denoted $\deg(v)$. We next give a formal definition of leverage centrality [1].

**Definition 1** (*Leverage Centrality*)**.** Leverage centrality is a measure of the relationship between the degree of a given node $v$ and the degree of each of its neighbors $v_i$, averaged over all neighbors $N_v$, and is defined as shown below:

$$l(v) = \frac{1}{\deg(v)} \sum_{v_i \in N_v} \frac{\deg(v) - \deg(v_i)}{\deg(v) + \deg(v_i)}.$$

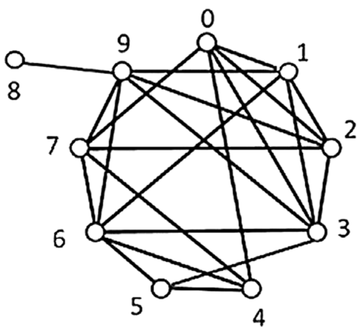
This property was used by Joyce et al. [1] in the analysis of functional magnetic resonance imaging (fMRI) data [1] and has also been applied to real-world networks including airline connections, electrical power grids, and coauthorship collaborations [2]. However despite these studies leverage centrality has yet to be explored from a mathematical standpoint. The formula gives a measure of the relationship between a vertex and its neighbors. A positive leverage centrality means that this vertex has influence over its neighbors, where as a negative leverage centrality indicates that a vertex is being influenced by its neighbors.

We begin with an elementary result involving the bounds of leverage centrality (Li et al. [2]).

**Lemma 2.** *Let $G$ be a graph with $n$ vertices. For any vertex $v$, $|l(v)| \leq 1 - \frac{2}{n}$. Furthermore, these bounds are tight in the cases of stars and complete graphs.*

We note that the bounds are also tight for regular graphs.

There exist graphs $G$ where the leverage centrality of all vertices is equal and where the leverage centrality of vertices is distinct. It is clear that if $G$ is a regular graph than $l(v) = 0$ for every $v \in G$. We give an example below of a graph that has distinct leverage centralities (see Fig. 1).




| | |
|---|---|
| 0 | $\frac{1}{5}\left(\frac{5-4}{5+4}\right) = \frac{1}{45}$ |
| 1 | $\frac{1}{5}\left(\frac{5-6}{5+6}\right) = -\frac{1}{55}$ |
| 2 | $0$ |
| 3 | $\frac{1}{6}\left(3\left(\frac{6-5}{6+5}\right) + \left(\frac{6-3}{6+3}\right)\right) = \frac{10}{99}$ |
| 4 | $\frac{1}{4}\left(\left(\frac{4-6}{4+6}\right) + 2\left(\frac{4-5}{4+5}\right) + \left(\frac{4-3}{4+3}\right)\right) = -\frac{22}{315}$ |
| 5 | $\frac{1}{3}2\left(\frac{3-6}{3+6}\right) = -\frac{2}{9}$ |
| 6 | $\frac{1}{6}\left(2\left(\frac{6-5}{6+5}\right) + \left(\frac{6-4}{6+4}\right) + \left(\frac{6-3}{6+3}\right)\right) = \frac{59}{495}$ |
| 7 | $\frac{1}{5}\left(2\left(\frac{5-6}{5+6}\right) + \left(\frac{5-4}{5+4}\right)\right) = -\frac{7}{495}$ |
| 8 | $\frac{1-6}{1+6} = -\frac{5}{7}$ |
| 9 | $\frac{1}{6}\left(3\left(\frac{6-5}{6+5}\right) + \left(\frac{6-1}{6+1}\right)\right) = \frac{38}{231}$ |

**Fig. 1.** A graph with distinct leverage centralities.

Intuitively one would think that the sum of the leverage centralities over a graph would be zero. This is in fact the case when a graph is regular. However, for non-regular graphs the sum of leverage centralities is negative. This arises since each edge between two vertices of different degrees contributes a negative amount to the sum of the leverage centralities. Let $G$ be the graph $K_3$ with a pendant edge (see Fig. 2).

Then $l(v_1) = \frac{1}{2}\left(\frac{2-3}{2+3}\right) + \frac{1}{2}\left(\frac{2-2}{2+1}\right)$; $l(v_2) = \frac{1}{3}\left(\frac{3-1}{3+1}\right) + \frac{1}{3}\left(\frac{3-2}{3+2}\right) + \frac{1}{3}\left(\frac{3-2}{3+2}\right)$; $l(v_3) = \frac{1}{1}\left(\frac{1-3}{1+3}\right)$; and $l(v_4) = \frac{1}{2}\left(\frac{2-3}{2+3}\right) + \frac{1}{2}\left(\frac{2-2}{2+2}\right)$. We can regroup the sum to be

$$\sum_{v_i \in G} l(v_i) = \frac{1}{2}\left(\frac{2-3}{2+3}\right) + \frac{1}{2}\left(\frac{2-2}{2+1}\right) + \frac{1}{3}\left(\frac{3-1}{3+1}\right) + \frac{1}{3}\left(\frac{3-2}{3+2}\right) + \frac{1}{3}\left(\frac{3-2}{3+2}\right)$$
$$+ \frac{1}{1}\left(\frac{1-3}{1+3}\right) + \frac{1}{2}\left(\frac{2-3}{2+3}\right) + \frac{1}{2}\left(\frac{2-2}{2+2}\right)$$

Please cite this article in press as: R. Vargas Jr, et al., A graph theoretic analysis of leverage centrality, AKCE International Journal of Graphs and Combinatorics (2017), http://dx.doi.org/10.1016/j.akcej.2017.05.001.