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Some notes on the isolate domination in graphs

Nader Jafari Rad

Department of Mathematics, Shahrood University of Technology, Shahrood, Iran

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Abstract

A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S is an *isolate dominating set* if the induced subgraph $G[S]$ has at least one isolated vertex. The *isolate domination number* $\gamma_0(G)$ is the minimum cardinality of an isolate dominating set of G. In this paper we study the complexity of the isolate domination in graphs, and obtain several bounds and characterizations on the isolate domination number, thus answering some open problems.

Keywords: Domination; Isolate domination; Total domination; Complexity

1. Introduction

For notation and graph theory terminology, we in general follow [1,2]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $\deg_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write $N(v)$ and $\deg(v)$ rather than $N_G(v)$ and $\deg_G(v)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A vertex of degree one is called a *leaf* and its unique neighbor a *support* vertex. A *pendant edge* is an edge which one of its end-points is a leaf. A *star* of order $n \geq 3$ is a tree that has precisely one vertex that is not a leaf. A *claw-free* graph is a graph with no induced subgraph isomorphic to a star of order 4. A *double-star* is a tree that has precisely two vertices that are not leaves. We refer $S(a, b)$ as a double-star which its central vertices have degree a and b, respectively. For a subset S of vertices of G we denote by $G[S]$ the subgraph of G *induced* by S. For two subsets of vertices X and Y of $V(G)$, we denote by $G[X, Y]$ the subgraph of G induced by $X \cup Y$. The *diameter* of a graph G, denoted by $diam(G)$, is the maximum distance between pairs of vertices of G. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle contained in G.

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E-mail address: n.jafarirad@gmail.com.

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A subset S of vertices of a graph G is a *dominating set* of G if every vertex in $V(G) - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. We refer a dominating set of cardinality $\gamma(G)$ as a $\gamma(G)$-set. A dominating set S in a graph with no isolated vertex is called a *total dominating set* of G if $G[S]$ has no isolated vertex. The *total domination number* $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. We refer a total dominating set of cardinality $\gamma_t(G)$ as a $\gamma_t(G)$-set. A total dominating set S in G is called an *efficient total dominating set* if the open neighborhoods of the vertices of S form a partition for $V(G)$. For a subset S of vertices of G, and a vertex $x \in S$, we say that a vertex $y \notin S$ is an *external private neighbor of x with respect to S* if $N(y) \cap S = \{x\}$. We denote by $epn(x, S)$ the set of all external private neighbors of x respect to S.

Hamid and Balamurugan [3] initiated the study of *isolate domination* in graphs. A dominating set S is an *isolate dominating set* if the induced subgraph $G[S]$ has at least one isolated vertex. The *isolate domination number* $\gamma_0(G)$ is the minimum cardinality of an isolate dominating set of G. The concept of isolate domination was further studied, for example in [4–9]. Hamid et al. [3] showed that for a cubic graph G, $\gamma(G) \leq \gamma_0(G) \leq \gamma(G) + 1$. They presented several bounds, and properties for the isolate domination number, and proposed the following problem(s).

Problems: (1) Characterize cubic graphs G with $\gamma_0(G) = \gamma(G) + 1$.
(2) Characterize graphs G with $\gamma_0(G) = \gamma(G)$, or $\gamma_0(G) = \frac{n}{2}$.
(3) Find bounds for $\gamma_0(G)$.

In this paper we first study the complexity of the isolate domination number in graph by showing that the decision problem for this variant is NP-complete, even when restricted to bipartite graphs. We then answer all of the above problems. We present several bounds, and characterizations for the isolate domination number in a graph.

In the following we state some known results that we need for the next. The *corona graph* of a graph G, denoted by GoK_1, is the graph obtained from G by adding a pendant edge to every vertex of G.

Theorem 1 ([1]). *For a graph G of order n with no isolated vertex, $\gamma(G) \leq \frac{n}{2}$, with equality if and only if each component of G is a C_4 or the corona HoK_1 for any connected graph H.*

Lemma 2 ([3]). *For paths and cycles of order n, $\gamma_0(P_n) = \gamma_0(C_n) = \lceil \frac{n}{3} \rceil$.*

Theorem 3 ([2]). *If G is a graph of order n and with no isolated vertex then $\gamma_t(G) \geq \frac{n}{\Delta(G)}$.*

2. Complexity

In this section we show that the decision problem for the isolate domination is NP-complete, even when restricted to bipartite graphs. We use a transformation from the 3-SAT problem. A *truth assignment* for a set U of Boolean variables is a mapping $t : U \to \{T, F\}$. A variable u is said to be *true* (or *false*) under t if $t(u) = T$ (or $t(u) = F$). If u is a variable in U, then u and $\overline{u}$ are *literals* over U. The literal u is true under t if and only if the variable u is true under t, and the literal $\overline{u}$ is true if and only if the variable u is false. A *clause* over U is a set of literals over U, and it is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over U is satisfiable if and only if there exists some truth assignment for U that simultaneously satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a satisfying truth assignment for $\mathcal{C}$. The 3-SAT problem is specified as follows.

3-SAT problem:

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set U of variables such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: Is there a truth assignment for U that satisfies all the clauses in C?

Note that the 3-SAT problem was proven to be NP-complete in [10]. Consider the following decision problem.

Isolate dominating set (IDS):

Instance: A graph $G = (V, E)$ and a positive integer k.

Question: Does G have an isolate dominating set of size at most k?

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