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AKCE International Journal of Graphs and Combinatorics

www.elsevier.com/locate/akcej

Packing chromatic number of certain fan and wheel related graphs

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Received 17 June 2015; accepted 20 May 2016

Available online xxxx

Abstract

The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k for which there exists a mapping $\pi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that any two vertices of color i are at distance at least $i + 1$. In this paper, we compute the packing chromatic number for certain fan and wheel related graphs.

Keywords: Packing chromatic number; Uniform n-fan split graph; Uniform n-wheel split graph

1. Introduction

Let G be a connected graph and k be an integer, $k \geq 1$. A packing k-coloring of a graph G is a mapping $\pi : V(G) \longrightarrow \{1, 2, \ldots, k\}$ such that any two vertices of color i are at distance at least $i + 1$. The packing chromatic number $\chi_\rho(G)$ of G is the smallest integer k for which G has packing k-coloring. The concept of packing coloring comes from the area of frequency assignment in wireless networks and was introduced by Goddard et al. [1] under the name broadcast coloring. It has several applications, such as, in resource placement and biological diversity. The term packing chromatic number was introduced by Brešar et al. [2].

Goddard et al. [1] proved that the packing coloring problem is NP-complete for general graphs and Fiala and Golovach [3] proved that it is NP-complete even for trees. It is proved that the packing coloring problem is solvable in polynomial time for graphs whose treewidth and diameter are both bounded [3] and for cographs and split graphs [1]. Sloper [4] studied a special type of packing coloring, called eccentric coloring and proved that the infinite 3-regular tree has packing chromatic number 7. For the infinite planar square lattice $\mathbb{Z}^2$, $10 \leq \chi_\rho(\mathbb{Z}^2) \leq 17$ [5,6]. The packing coloring of distance graphs was studied in [7,8]. For the infinite hexagonal lattice $\mathbb{H}$, $\chi_\rho(\mathbb{H}) = 7$ [2].

Argiroffo et al. [9,10] proved that the packing coloring problem is solvable in polynomial time for the class of $(q, q-4)$ graphs, partner limited graphs and for an infinite subclass of lobsters, including caterpillars. It is proved in [11,12] that the infinite, planar triangular lattice and the three dimensional square lattice have unbounded packing chromatic number. In this paper, we study the packing chromatic number of certain fan and wheel related graphs.

Peer review under responsibility of Kalasalingam University.
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http://dx.doi.org/10.1016/j.akcej.2016.11.001

2. Main results

Let G_1 and G_2 be vertex disjoint graphs with $|V(G_1)| = n_1$, $|E(G_1)| = m_1$, $|V(G_2)| = n_2$ and $|E(G_2)| = m_2$.

Definition 2.1. The union of G_1 and G_2 is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. It is denoted by $G_1 \cup G_2$. So, $G_1 \cup G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2$ edges.

Definition 2.2. The sum or join of G_1 and G_2 is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 with G_2. It is denoted by $G_1 + G_2$. So, $G_1 + G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2 + n_1 n_2$ edges.

Definition 2.3. A Fan graph F_n is defined as the graph $K_1 + P_n$, where K_1 is the singleton graph and P_n is the path on n vertices.

Definition 2.4. The wheel W_{n+1} is defined as the graph $K_1 + C_n$, where K_1 is the singleton graph and C_n is the cycle graph on n vertices.

Definition 2.5 (*[13]*)**.** A uniform n-fan split graph SF_n^r contains a star S_{n+1} with hub at x such that the deletion of the n edges of S_{n+1} partitions the graph into n independent fans $F_r^i = P_r^i + K_1$, $1 \le i \le n$ and an isolated vertex. See Fig. 1.

Theorem 2.6. *For the uniform n-fan split graph SF_n^r, $n \ge 4$, $r \ge 5$, we have $\chi_\rho(SF_n^r) \ge 3 + n[r - \lceil \frac{r}{2} \rceil - 1]$.*

Proof. Let F_r^i, $1 \le i \le n$ be the fans of SF_n^r. Let $V(SF_n^r) = \{w_j^i, w_i, x : 1 \le i \le n, 1 \le j \le r\}$, where w_i is the hub of F_r^i and x is the hub of S_{n+1}. Since the diameter of SF_n^r is 4, colors greater than 3 can be assigned to only one vertex of SF_n^r.
Fact 1: If color 3 is assigned to vertex x, no other vertex of SF_n^r can receive color 3 because $d(x, w_j^i) = 2$ and $d(x, w_i) = 1$, $1 \le i \le n$, $1 \le j \le r$. Similarly, if color 3 is assigned to any vertex w_i, no other vertex of SF_n^r can receive color 3. And also, if color 3 is assigned to any vertex w_j^i of any F_r^i and since $\text{diam}(F_r^i) = 2$, no other vertex of chosen F_r^i can receive 3. There are n fans in SF_n^r. Since $d(w_j^i, w_m^l) = 4$, $i \ne l$, $1 \le i, l \le n$, $1 \le j, m \le r$, at most n vertices receive color 3. Thus, the maximum number of vertices that can receive color 3 is n.
Fact 2: Since $d(w_j^i, w_m^l) = 4$, $i \ne l$, $1 \le i, l \le n$, $1 \le j, m \le r$ and $d(w_i, w_m^l) = 3$, $i \ne l$, $1 \le i, l \le n$, $1 \le m \le r$, assigning color 2 to a vertex w_j^i or w_i, at most n vertices can receive 2. Thus, the maximum number of vertices that can receive color 2 is n.
Fact 3: If color 1 is assigned to any vertex w_i, at most $(n-1)\lceil \frac{r}{2} \rceil + 1$ vertices can receive 1. But, if color 1 is assigned to vertex x and alternative vertices of F_r^i with 1, at most $n\lceil \frac{r}{2} \rceil + 1$ vertices can receive 1. Thus, the maximum number of vertices that can receive color 1 is $n\lceil \frac{r}{2} \rceil + 1$.

There are $nr + n + 1$ vertices in SF_n^r and at most $n\lceil \frac{r}{2} \rceil + 1 + n + n$ vertices receive color 1, 2 and 3. Thus, at least $nr + n + 1 - [n + n + n\lceil \frac{r}{2} \rceil + 1] = n[r - \lceil \frac{r}{2} \rceil - 1]$ vertices should receive distinct colors starting from 4 to $3 + n[r - \lceil \frac{r}{2} \rceil - 1]$. Thus, $\chi_\rho(SF_n^r) \ge 3 + n[r - \lceil \frac{r}{2} \rceil - 1]$.

We give an algorithm to color the uniform n-fan split graph SF_n^r and prove that the bound is sharp.

Procedure PACKING COLORING SF_n^r, $n \ge 4$, $r \ge 5$
Input: A uniform n-fan split graph SF_n^r
Algorithm:
Step 1: Color the vertices w_{2j-1}^i, $1 \le i \le n$, $1 \le j \le \lceil \frac{r}{2} \rceil$ of F_r^i by 1.
Step 2: Color the vertices w_{2j}^i, $1 \le i \le n$, $1 \le j \le 2$ of F_r^i by $(1 + j)$.
Step 3: Color the hub vertex x by 1.
Step 4: Color remaining vertices of SF_n^r with distinct colors starting from 4 to $3 + n[r - \lceil \frac{r}{2} \rceil - 1]$.
Output: A packing $3 + n[r - \lceil \frac{r}{2} \rceil - 1]$-coloring of SF_n^r.
Proof of Correctness: The diameter of F_r^i is 2. Coloring the vertices w_{2j-1}^i, $1 \le i \le n$, $1 \le j \le \lceil \frac{r}{2} \rceil$ of any F_r^i by 1, at most $\lceil \frac{r}{2} \rceil$ vertices receive color 1. There are n fans in SF_n^r and since $d(w_j^i, w_m^l) = 4$, $i \ne l$, $1 \le i, l \le n$, $1 \le j, m \le r$, at most $n\lceil \frac{r}{2} \rceil$ vertices receive color 1. Since $\text{diam}(F_r^i) = 2$, colors greater than 1 cannot be used more than