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On clique convergence of graphs

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Abstract

Let G be a graph and $\mathcal{K}_G$ be the set of all cliques of G, then the clique graph of G denoted by $K(G)$ is the graph with vertex set $\mathcal{K}_G$ and two elements $Q_i, Q_j \in \mathcal{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$. Iterated clique graphs are defined by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ for $n > 0$. In this paper we prove a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$, give a partial characterization for clique divergence of the join of graphs and prove that if G_1, G_2 are Clique-Helly graphs different from K_1 and $G = G_1 \Box G_2$, then $K^2(G) = G$.

Keywords: Maximal clique; Clique graph; Graph operator

1. Introduction

Given a simple graph $G = (V, E)$, not necessarily finite, a clique in G is a maximal complete subgraph in G. Let G be a graph and $\mathcal{K}_G$ be the set of all cliques of G, then the clique graph operator is denoted by K and the clique graph of G is denoted by $K(G)$, where $K(G)$ is the graph with vertex set $\mathcal{K}_G$ and two elements $Q_i, Q_j \in \mathcal{K}_G$ form an edge if and only if $Q_i \cap Q_j \neq \emptyset$. Clique graph was introduced by Hamelink in 1968 [1]. Iterated clique graphs are defined by $K^0(G) = G$, and $K^n(G) = K(K^{n-1}(G))$ for $n > 0$ (see [2–4]).

Definition 1.1. A graph G is said to be K-periodic if there exists a positive integer n such that $G \cong K^n(G)$ and the least such integer is called the K-periodicity of G, denoted K-per (G).

Definition 1.2. A graph G is said to be K-Convergent if $\{K^n(G) : n \in \mathbb{N}\}$ is finite, otherwise it is K-Divergent (see [5]).

Definition 1.3. A graph H is said to be K-root of a graph G if $K(H) = G$.

If G is a clique graph then one can observe that, the set of all K-roots of G is either empty or infinite.

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Definition 1.4 (*[3]*)**.** A graph G is a Clique-Helly Graph if the set of cliques has the Helly-Property. That is, for every family of pairwise intersecting cliques of the graph, the total intersection of all these cliques should be non-empty also.

Definition 1.5. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be the two graphs. Then their join $G_1 + G_2$ is obtained by adding all possible edges between the vertices of G_1 and G_2.

Definition 1.6. The Cartesian product of two graphs G and H, denoted $G \Box H$, is a graph with vertex set $V(G \Box H) = V(G) \times V(H)$, i.e., the set $\{(g, h) | g \in G, h \in H\}$. The edge set of $G \Box H$ consists of all pairs $[(g_1, h_1), (g_2, h_2)]$ of vertices with $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or $g_1 = g_2$ and $[h_1, h_2] \in E(H)$ (see [6] page no 3).

2. Results

One can observe that the clique graph of a complete graph and star graph are always complete. Let G be a graph with n vertices and having a vertex of degree $n - 1$, then the clique graph of G is also complete.

Theorem 2.1. *Let G_1, G_2 be two graphs and $G = G_1 + G_2$, then X is a clique in G_1 and Y is a clique in G_2 if and only if $X + Y$ is a clique in $G_1 + G_2$.*

Proof. Let $G = G_1 + G_2$ and X be a clique in G_1 and Y be a clique in G_2. Suppose that $X + Y$ is not a maximal complete subgraph in $G_1 + G_2$, then there is a maximal complete subgraph (clique) Q in $G_1 + G_2$ such that $X + Y$ is a proper subgraph of Q. Since $X + Y$ is a proper subgraph of Q, there is a vertex v in Q which is not in $X + Y$ and v is adjacent to every vertex of $X + Y$, then by the definition of $G_1 + G_2$, v should be in either G_1 or G_2. Suppose v is in G_1, then the induced subgraph of $V(X) + \{v\}$ is complete in G_1, which is a contradiction as X is maximal. Therefore $X + Y$ is the maximal complete subgraph (clique) in $G_1 + G_2$.

Conversely, let Q is a clique in $G_1 + G_2$. Suppose that $Q \neq X + Y$ where X is a clique in G_1 and Y is a clique in G_2. If $Q \cap G_1 = \emptyset$, then Q is a subgraph of G_2. This implies that Q is a clique in G_2 as Q is a clique in G. Let v be a vertex of G_1. Then by the definition of $G_1 + G_2$, one can observe that the induced subgraph of $V(Q) \cup \{v\}$ is complete in G, which is a contradiction as Q is a maximal complete subgraph. Therefore $Q \cap G_1 \neq \emptyset$. Similarly we can prove that $Q \cap G_2 \neq \emptyset$. Let X be the induced subgraph of G with vertex set $V(Q) \cap V(G_1)$ and Y be the induced subgraph of G with vertex set $V(Q) \cap V(G_2)$, then $Q = X + Y$. Since Q is a maximal complete subgraph of G, X and Y should be maximal complete subgraphs in G_1 and G_2 respectively. Otherwise, if X is not a maximal complete subgraph in G_1 then there is a maximal complete subgraph X' in G_1 such that X is subgraph of X', and this implies that $X + Y$ is a subgraph of $X' + Y$ and $X' + Y$ is complete, which is a contradiction. Therefore X and Y are maximal complete subgraphs (cliques) in G_1 and G_2 respectively. ■

Corollary 2.2. *Let G_1, G_2 be two graphs and $G = G_1 + G_2$. If n, m are the number of cliques in G_1, G_2 respectively, then G has nm cliques.*

Proof. Let $G = G_1 + G_2$, $\mathcal{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$ be the set of all cliques of G_1 and $\mathcal{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$ be the set of all cliques of G_2. Then by Theorem 2.1 it follows that $\mathcal{K}_G = \{X_i + Y_j : 1 \le i \le n, 1 \le j \le m\}$ is the set of all cliques of G. Since G_1 has n, G_2 has m number of cliques, $G_1 + G_2$ has nm number of cliques. ■

In the following result we give a necessary and sufficient condition for a clique graph $K(G)$ to be complete when $G = G_1 + G_2$.

Theorem 2.3. *Let G_1, G_2 be two graphs. If $G = G_1 + G_2$, then $K(G)$ is complete if and only if either $K(G_1)$ is complete or $K(G_2)$ is complete.*

Proof. Let $G = G_1 + G_2$ and $K(G)$ be complete. Suppose that neither $K(G_1)$ nor $K(G_2)$ is complete, then there exist two cliques X, X' in G_1 and two cliques Y, Y' in G_2 such that $X \cap X' = \emptyset$ and $Y \cap Y' = \emptyset$. By Theorem 2.1 it follows that $X + Y$, $X' + Y'$ are cliques in G. Since $X \cap X'$ and $Y \cap Y'$ are empty, it follows that $\{X + Y\} \cap \{X' + Y'\} = \emptyset$, which is a contradiction as $K(G)$ is complete.

Conversely, suppose that $K(G_1)$ is complete and $\mathcal{K}_{G_1} = \{X_1, X_2, \ldots, X_n\}$, $\mathcal{K}_{G_2} = \{Y_1, Y_2, \ldots, Y_m\}$. By Corollary 2.2, it follows that G has exactly nm number of cliques. Let $\mathcal{K}_G = \{Q_{ij} : Q_{ij} = X_i + Y_j$ for $i =$