ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

AKCE International Journal of Graphs and Combinatorics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

AKCE International Journal of Graphs and Combinatorics

www.elsevier.com/locate/akcej

Decomposing certain equipartite graphs into sunlet graphs of length $2p$

Abolape D. Akwu[a], Deborah Olayide A. Ajayi[b,*]

[a] Department of Mathematics, Federal University of Agriculture, Makurdi, Nigeria
[b] Department of Mathematics, University of Ibadan, Ibadan, Nigeria

Received 29 February 2016; received in revised form 23 May 2016; accepted 18 August 2016

Abstract

For any integer $r \geq 3$, we define the sunlet graph of order $2r$, denoted L_{2r}, as the graph consisting of a cycle of length r together with r pendant vertices, each adjacent to exactly one vertex of the cycle. In this paper, we give necessary and sufficient conditions for decomposing the lexicographic product of the complete graph and the complete graph minus a 1-factor, with complement of the complete graph K_m, (that is $K_n \otimes \bar{K}_m$ and $K_n - I \otimes \bar{K}_m$, respectively) into sunlet graphs of order twice a prime.

Keywords: Graph decomposition; Corona graph; Lexicographic product; Equipartite graph; Sunlet graph

1. Introduction and notations

All graphs considered here are simple and finite. Let C_r, K_n, $\bar{K}_n$ and $K_{n,n}$ denote a cycle of length r, the complete graph on n vertices, its complement and a complete bipartite graph, respectively. Also, let L_{2p} stand for the sunlet graph (or corona graph) which is a graph that consists of a cycle and an edge terminating in a vertex of degree one attached to each vertex of cycle C_p. A graph G is said to be *decomposable* into a graph H or H *decomposes* G, if G can be written as the union of edge-disjoint copies of H so that every edge in G belongs to one and only one copy of H. We write $G \cong H \oplus H \oplus \cdots \oplus H$, or simply $H|G$ if G is decomposable into H. Also, $H|G$ means *decomposition* of G into copies of H.

We denote by $G \otimes H$ the *lexicographic product* of graphs G and H, which is obtained by replacing every vertex of G by a copy of H and every edge of G by the complete bipartite graph $K_{|H|,|H|}$. The graph $K_n \otimes \bar{K}_m$ is isomorphic to the complete n-partite graph in which each partite set has exactly m vertices. The graph $C_r \otimes \bar{K}_m$ is an equipartite graph, with the degree of any vertex being $2m$ and the total number of edges is rm^2. The graph $(K_n - I)$ is a complete graph minus a 1-factor and its lexicographic product with $\bar{K}_m$, $(K_n - I) \otimes \bar{K}_m$, is an equipartite graph.

Peer review under responsibility of Kalasalingam University.
* Corresponding author.
E-mail addresses: abolapeyemi@yahoo.co.uk (A.D. Akwu), adelaideajayi@yahoo.com, olayide.ajayi@ui.edu.ng (D.O.A. Ajayi).

http://dx.doi.org/10.1016/j.akcej.2016.08.001

Please cite this article in press as: A.D. Akwu, D.O.A. Ajayi, Decomposing certain equipartite graphs into sunlet graphs of length $2p$, AKCE International Journal of Graphs and Combinatorics (2016), http://dx.doi.org/10.1016/j.akcej.2016.08.001

A great deal of research has been done on the study of decomposition of complete graphs into cycles (see for example [1–3]).

Obvious necessary conditions for the existence of a k-cycle decomposition of a simple connected graph G is that G has at least k vertices (or trivially, just one vertex), the degree of every vertex in G is even and the total number of edges in G is a multiple of the cycle length k. These conditions have been shown to be sufficient in the case that G is the complete graph, K_n, or the complete graph minus a 1-factor, $K_n - I$ [1,3].

The study of cycle decomposition of $K_n \otimes \bar{K}_m$ was initiated by Hoffman et al. [4]. The necessary and sufficient conditions for the existence of a C_p-decomposition of $K_n \otimes \bar{K}_m$, where $p \geq 5$ (p is prime), was obtained by Manikandan and Paulraja [5,6]. Similarly, when $p \geq 3$ is a prime, the necessary and sufficient conditions for the existence of a C_{2p}-decomposition of $K_n \otimes \bar{K}_m$ were obtained by Smith [7]. For a prime number $p \geq 3$, it was proved by Smith [8] that C_{3p}-decomposition of $K_n \otimes \bar{K}_m$ exists if the obvious necessary conditions are satisfied. In [9], Anitha and Lekshmi proved that the complete graph K_n for n even has a decomposition into sunlet graph L_n.

The necessary condition for the equipartite graphs $(K_n - I) \otimes \bar{K}_m$ and $K_n \otimes \bar{K}_m$ to be decomposed into L_{2p} is that the number of edges in $(K_n - I) \otimes \bar{K}_m$ and $K_n \otimes \bar{K}_m$ must be a multiple of $2p$. In this paper, we consider the decomposition of $C_r \otimes \bar{K}_m$, $K_n \otimes \bar{K}_m$ and $(K_n - I) \otimes \bar{K}_m$ into L_{2p}, p a prime number and prove among other results, that the necessary condition is sufficient that is:

1. Let $m \geq 2$ and $n > 3$ be even integers, p an odd prime. The graph $(K_n - I) \otimes \bar{K}_m$ admits a decomposition into sunlet graphs L_{2p} if and only if the obvious necessary condition $\frac{1}{2}n(n-2)m^2 \equiv 0 (\mathrm{mod}\ 2p)$ is satisfied.
2. Let $m \geq 2$ and $n \geq 3$ be even and odd integers respectively, p an odd prime. The graph $K_n \otimes \bar{K}_m$ admits a decomposition into sunlet graphs L_{2p} if and only if the obvious necessary condition $\frac{1}{2}n(n-1)m^2 \equiv 0 (\mathrm{mod}\ 2p)$ is satisfied.

2. Proof of results

To prove the results, we need the following.

Lemma 2.1. *For $r \geq 3$, $L_{2r} | C_r \otimes \bar{K}_2$.*

Proof. From the definition of the graph $C_r \otimes \bar{K}_2$, each vertex x_i in C_r is replaced by a pair of two independent vertices $x_{i,1}, x_{i,2}$ and each edge $x_i x_j$ is replaced by four edges $x_{i,1}x_{j,1}, x_{i,1}x_{j,2}, x_{i,2}x_{j,1}, x_{i,2}x_{j,2}$. First we construct two base cycles C_r^1 and C_r^2 as follows:

$$C_r^1 = x_{1,1}x_{2,1}x_{3,1}...x_{r,1}$$

and

$$C_r^2 = x_{1,2}x_{2,2}x_{3,2}...x_{r,2}.$$

Therefore we have two cycles C_r^1 and C_r^2. Define a mapping ϕ by $\phi(x_{i,1}) = x_{i+1,2}$ and $\phi(x_{i,2}) = x_{i+1,1}$, where the sum on the first suffix is taken modulo r. The vertices $\phi(x_{i,1})$ and $\phi(x_{i,2})$ are pendant vertices. Attach each pendant vertex $\phi(x_{i,1})$ to each vertex $x_{i,1}$ in C_r^1 and $\phi(x_{i,2})$ to each vertex $x_{i,2}$ in C_r^2. The cycles C_r^1, C_r^2 with pendant vertices $\phi(x_{i,1})$ and $\phi(x_{i,2})$ attached respectively give two sunlet graphs with $2r$ vertices. Hence, $C_r \otimes \bar{K}_2$ can be decomposed into two copies of L_{2r}. □

Lemma 2.2. *The graph $L_{2r} \otimes \bar{K}_l$ is decomposable into l^2 copies of L_{2r} for any positive integer l and $r > 2$.*

Proof. Let the vertices of sunlet graphs on $2r$ vertices L_{2r} be $\{1, 2, \ldots, r, 1', 2', \ldots, r'\}$ where i' is the pendant vertex of L_{2r} connected with the vertex I of the cycle in L_{2r}. Then the vertex of L_{2r} wreath compliment of complete graph on l vertices are $\{(p, a), (p', a) \mid p, p' = 1, 2, \ldots, r \text{ and } a = 1, 2, \ldots, l\}$. Then form the sunlet graphs $L_{2r}^1, \ldots, L_{2r}^{l^2}$ from $L_{2r} \otimes \bar{K}_l$ as follows:

Construct an $l \times l$ Latin square. For each of the l^2 element, we can form an r-cycle, C_r. If r is even, C_r is of the form $(1, u), (2, v), (3, u), \ldots, (r-1, u), (r, \alpha)$ and if r is odd, C_r is of the form $(1, u), (2, v), (3, u), \ldots, (r-1, v), (r, \alpha)$, where u is the row, v is the column and α $(1 \leq \alpha \leq l)$ is the entry in the Latin square.

Next form the sunlet graph L_{2r} from each of the l^2 r-cycles, as follows: