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Characterizing minimal point set dominating sets

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Dedicated to sweet memories of Dr. B.D Acharya

Abstract

A set D of vertices in a graph $G = (V, E)$ is said to be a point-set dominating set (or, in short, psd-set) of G if for every subset S of $V - D$ there exists a vertex $v \in D$ such that the subgraph $\langle S \cup \{v\} \rangle$ is connected; the set of all psd-sets of G will be denoted $\mathfrak{D}_{ps}(G)$. The point-set domination number of a graph denoted by $\gamma_p(G)$ is the minimum cardinality of a psd-set of G. We obtain a lower bound for $\gamma_p(G)$ and characterize graphs which attain this bound. A psd-set D of a graph G is *minimal* if no proper subset of D is a psd-set of G. In this paper, we give a general characterization of psd-sets which are minimal. Also, in the case of separable graphs, we obtain more transparent and structure specific characterizations of minimal psd-sets.

Keywords: Domination; Point-set domination; Minimal point-set dominating set

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to West [1].

A set D of vertices of a graph G is said to be a *dominating set* [2–4] (or, in short, dom-set) if every vertex $v \in V - D$ is adjacent to some vertex $u \in D$. The set of all dominating sets in G will be denoted by $\mathfrak{D}(G)$. The domination number of a graph G denoted by $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A set D of vertices in a graph G is a *point-set dominating set* (or, in short, psd-set) of G if for every subset $S \subseteq V - D$ there exists $v \in D$ such that the subgraph $\langle S \cup \{v\} \rangle$ is connected. The set of all psd-sets in G will be denoted by $\mathfrak{D}_{ps}(G)$. The point-set domination number of a graph G, denoted by $\gamma_p(G)$, is the minimum cardinality

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of a psd-set of G. The set of all psd-sets of G with cardinality $\gamma_p(G)$ is denoted by $\mathfrak{D}^o_{ps}(G)$. Further for any graph G, a psd-set of G with exactly $\gamma_p(G)$ elements will be called a γ_p-set (or, a minimum psd-set) of G or, more simply, a $\gamma_p(G)$-set.

Clearly every psd-set of a graph G must be a dominating set of G but converse need not be true. Hence we have

$$\gamma(G) \leq \gamma_p(G) \quad \text{and} \quad \mathfrak{D}_{ps}(G) \subseteq \mathfrak{D}(G). \tag{1}$$

Lemma 1.1 (*[5]*)**.** *Let G be any graph and D be any psd-set of G. Then, $\langle V - D\rangle$ is a proper subgraph of a component H of G. Conversely, if H is a component of G then a psd-set of H, together with the vertex sets of all other components of G, forms a psd-set of G.*

Theorem 1.2 (*[5]*)**.** *Let G be a graph of order n, and $\mathfrak{C}_G$ denote the set of its components. Then*

$$\gamma_p(G) = n - \max_{H\in\mathfrak{C}_G} (|V(H)| - \gamma_p(H)).$$

We observe from Theorem 1.2 that to find the point set domination number of a graph G, we need to find the point set domination number of each of its components. Hence, without loss of generality, we may assume that G is connected which we shall do hither forth.

The following two interesting observations made in [6] are easy consequences of the definition of psd-sets and are useful while working on psd-sets.

Proposition 1 (*[6]*)**.** *Let G be a graph and D be a psd-set. Then $d(u, v) \leq 2$ for all $u, v \in V - D$. Also, if G is a graph with maximum degree $\Delta(G)$ and order n, then*

$$\gamma_p(G) \leq n - \Delta(G).$$

Walikar, Acharya and Sampathkumar [7] gave the following theorem that gives a lower bound for the domination number.

Theorem 1.3 (*[7]*)**.** *Let G be any graph of order n and maximum degree Δ, then*

$$\frac{n}{1+\Delta} \leq \gamma(G). \tag{2}$$

It follows from (1) and (2) that for any graph G of order n and maximum degree Δ,

$$\frac{n}{1+\Delta} \leq \gamma_p(G). \tag{3}$$

In the next theorem we characterize extremal graphs attaining the bound in (3).

Theorem 1.4. *Let G be any graph of order n and maximum degree Δ. Then*

$$\frac{n}{1+\Delta} = \gamma_p(G) \text{ if and only if } \Delta = n - 1. \tag{4}$$

Proof. Let equality in (4) hold and D be a $\gamma_p(G)$-set of G. Then $|V - D| = \Delta|D|$. As every psd-set is a dom-set, we have

$$|V - D| \leq \sum_{u\in D} d(u) \leq \sum_{u\in D} \Delta = \Delta|D| = |V - D|.$$

Hence $\sum_{u\in D}\big(\Delta - d(u)\big) = 0$ which implies $\Delta = d(u)$ for each $u \in D$.

Since $|V - D| = \Delta|D|$ and $\Delta = d(u)$ for all $u \in D$, D is independent and $N(v) \cap N(u) = \phi$ for all v, u $(v \neq u) \in D$. We claim that $|D| = 1$. If not, let u and v be two distinct vertices in D. Then D being a psd-set, $N(v) \cup \{u\} \subseteq N(x)$ for each $x \in N(u)$. Consequently, $d(x) \geq |N(v)| + 1 = \Delta + 1$, a contradiction. Thus $|D| = 1$ and $\Delta = n - 1$.

Conversely, if $\Delta = n - 1$, then equality in (4) is satisfied trivially. □