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Note

An interesting q-series related to the 4th symmetrized rank function

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ABSTRACT

This paper presents the method to utilizing the s-fold extension of Bailey's lemma to obtain spt-type functions related to the symmetrized rank function $\eta_{2k}(n)$. We provide the $k = 2$ example, but clearly illustrate how deep connections between higher-order spt functions exist for any integer $k > 1$, and provide several directions for possible research. In particular, we present why the function $spt_M^*(n)$, the total number of appearances of the smallest parts of partitions where parts greater than the smallest plus M do not occur, is an spt function that appears to have central importance. We also make note about extending spt-type functions to partition pairs.

1. Introduction and main results

As usual [9], set $(X)_n = (X; q)_n := \prod_{0\le k\le n-1}(1 - Xq^k)$, and define $(X)_\infty = (X; q)_\infty := \lim_{n\to\infty}(X; q)_n$. In [5], Andrews constructed an s-fold extension of Bailey's lemma, and obtained many interesting higher dimensional identities. In [3], we find that $spt(n)$, the number of appearances of the smallest parts in the number of partitions of n, satisfies the equation $spt(n) = np(n) - \frac{1}{2}N_2(n)$. Here $N_k(n) = \sum_{m\in\mathbb{Z}} m^k N(m, n)$, where $N(m, n)$ is the number of partitions of n with rank m [1]. Here we have kept the usual notation for the number of unrestricted partitions of n, $p(n)$ [1].

The results of Garvan [8] extended Andrews' important smallest part identity, allowing for connections between higher-order spt functions with the symmetrized rank function (defined by Eq. (2.6) and [2]) $\eta_{2k}(n)$, for any integer $k > 1$. Other smallest part functions with further restrictions on parts have also been noted in Patkowski [13,14]. As it turns out, we are able to show there are other spt functions for each $k > 1$ that follow from the k-fold Bailey lemma. In this sense, the order of the smallest part functions in Garvan's paper [8] correspond to the number of folds in the Bailey lemma in the sense of [5]. This paper will provide the $k = 2$ example, which will therefore require only the 2-fold Bailey lemma.

Theorem 1.1. *We have*

$$\sum_{n_1\ge 1}\frac{q^{n_1}}{(1-q^{n_1})^2(q^{n_1+1})_\infty}\sum_{n_2\ge 1}\frac{q^{n_2}}{(1-q^{n_2})^2(q^{n_2+1})_{n_1}}$$
$$=\frac{1}{(q)_\infty}\left(\sum_{n\ge 1}\frac{nq^n}{1-q^n}\right)^2+\frac{1}{(q)_\infty}\sum_{\substack{n\in\mathbb{Z}\\ n\ne 0}}\frac{(-1)^n q^{3n(n+1)/2}}{(1-q^n)^4}. \tag{1.1}$$

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Define the sequence $SPT^{+}(n)$, to be

$$\sum_{n\geq 1} SPT^{+}(n)q^{n}=\sum_{n_1\geq 1}\frac{q^{n_1}}{(1-q^{n_1})^2(q^{n_1+1})_{\infty}}\sum_{n_2\geq 1}\frac{q^{n_2}}{(1-q^{n_2})^2(q^{n_2+1})_{n_1}}.$$

$SPT^{+}(n)$ counts the number of partition pairs (λ,μ) with weight $w(\lambda,\mu)$ where λ is a partition, μ is a partition with the difference between its largest parts and the smallest parts bounded by the smallest part of λ, and $w(\lambda,\mu)$ is the product of the number of smallest parts of λ and the number of smallest parts of μ. Paraphrasing our Theorem 1.1 in terms of partitions and the rank generating function, we may now claim the following corollary, which we state as our next theorem.

Theorem 1.2. *For each natural number n, we have,*

$$SPT^{+}(n)=\frac{5}{72}M_4(n)-\frac{1}{6}n^2p(n)+\frac{1}{36}np(n)-\eta_4(n). \tag{1.2}$$

A natural consequence of Theorem 1.2 and known congruences for $p(n)$, $M_4(n)$, and $\eta_4(n)$ is the following result.

Theorem 1.3. *We have,*

$$SPT^{+}(7n)\equiv 0 \pmod 7, \tag{1.3}$$

$$SPT^{+}(11n)\equiv 0 \pmod{11}. \tag{1.4}$$

As an example, the partition pairs of 3 under this condition are (λ's in the first component, μ's in the second) $(3,1)$, $(2+1,1)$, $(1+1+1,1)$, $(2,2)$, $(2,1+1)$, $(1+1,2)$, $(1+1,1+1)$, $(1,3)$, $(1,2+1)$, $(1,1+1+1)$. The products of the number of smallest parts of the components yield $1+1+3+1+2+2+4+1+1+3=19$.

2. Proofs of theorems

In order to prove our results we need to state the $s=2$ case of the s-fold Bailey lemma given in [5].

2-**fold Bailey's Lemma** [5, Theorem 1] *We define a pair of sequences $(\alpha_{n_1,n_2},\beta_{n_1,n_2})$ to be a 2-fold Bailey pair with respect to a_j, $j=1,2$, if*

$$\beta_{n_1,n_2}=\sum_{r_1\geq 0}^{n_1}\sum_{r_2\geq 0}^{n_2}\frac{\alpha_{r_1,r_2}}{(a_1q;q)_{n_1+r_1}(q;q)_{n_1-r_1}(a_2q;q)_{n_2+r_2}(q;q)_{n_2-r_2}}. \tag{2.1}$$

Furthermore, we have

$$\sum_{n_1\geq 0}^{\infty}\sum_{n_2\geq 0}^{\infty}(x)_{n_1}(y)_{n_1}(z)_{n_2}(w)_{n_2}(a_1q/xy)^{n_1}(a_2q/zw)^{n_2}\beta_{n_1,n_2}$$
$$=\frac{(a_1q/x)_{\infty}(a_1q/y)_{\infty}(a_2q/z)_{\infty}(a_2q/w)_{\infty}}{(a_1q)_{\infty}(a_1q/xy)_{\infty}(a_2q)_{\infty}(a_2q/zw)_{\infty}}\sum_{n_1\geq 0}^{\infty}\sum_{n_2\geq 0}^{\infty}\frac{(x)_{n_1}(y)_{n_1}(z)_{n_2}(w)_{n_2}(a_1q/xy)^{n_1}(a_2q/zw)^{n_2}\alpha_{n_1,n_2}}{(a_1q/x)_{n_1}(a_1q/y)_{n_1}(a_2q/z)_{n_2}(a_2q/w)_{n_2}}. \tag{2.2}$$

In a paper by Joshi and Vyas [10], we find they have studied the special case of the s-fold extension. In our interest we will consider their use of the 2-fold extension of Bailey's lemma where they have chosen the α_{n_1,n_2} to be $\alpha_{n_1,n_2}=\alpha_n$, when $n_1=n_2=n$, and 0 otherwise. In particular, we find [10] that $(\alpha_{n_1,n_2},\beta_{n_1,n_2})$ is a 2-fold Bailey pair with respect to $a_j=1$, $j=1,2$, where $\alpha_{0,0}=1$,

$$\alpha_{n_1,n_2}=\begin{cases}(-1)^n q^{n(3n-1)/2}(1+q^n), & \text{if } n_1=n_2=n,\\ 0, & \text{otherwise,}\end{cases} \tag{2.3}$$

and

$$\beta_{n_1,n_2}=\frac{1}{(q)_{n_1}(q)_{n_2}(q)_{n_1+n_2}}. \tag{2.4}$$

Proof of Theorem 1.1. Differentiating (2.2) in the same manner as in [14] but appealing to all variables x, y, z, w, when setting equal to 1 (after setting $a_1=a_2=1$),

$$\sum_{n_1\geq 1}\sum_{n_2\geq 1}(q)_{n_1-1}^2(q)_{n_2-1}^2\beta_{n_1,n_2}q^{n_1+n_2}=\left(\sum_{n\geq 1}\frac{nq^n}{1-q^n}\right)^2+\sum_{n_1\geq 1}\sum_{n_2\geq 1}\frac{\alpha_{n_1,n_2}q^{n_1+n_2}}{(1-q^{n_1})^2(1-q^{n_2})^2}. \tag{2.5}$$