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A characterization for the neighbor-distinguishing total chromatic number of planar graphs with $\Delta = 13$

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ABSTRACT

The neighbor-distinguishing total chromatic number $\chi''_a(G)$ of a graph G is the smallest integer k such that G can be totally colored using k colors with a condition that any two adjacent vertices have different sets of colors. In this paper, we give a sufficient and necessary condition for a planar graph G with maximum degree 13 to have $\chi''_a(G) = 14$ or $\chi''_a(G) = 15$. Precisely, we show that if G is a planar graph of maximum degree 13, then $14 \leq \chi''_a(G) \leq 15$; and $\chi''_a(G) = 15$ if and only if G contains two adjacent 13-vertices.

1. Introduction

All graphs considered in this paper are finite and simple. Let G be a plane graph with vertex set $V(G)$, edge set $E(G)$, face set $F(G)$, minimum degree $\delta(G)$ and maximum degree $\Delta(G)$ (for short, Δ). An *element* of G is a member of $V(G)\cup E(G)\cup F(G)$. Two elements are *adjacent* if they are either adjacent to or incident with each other in the classical sense. For positive integers p, q with $p \leq q$, let $[p, q]$ denote the set of all integers between p and q.

A *total k-coloring* of a graph G is a mapping $\phi : V(G) \cup E(G) \to \{1, 2, \dots, k\}$ such that no two adjacent elements receive same color. The *total chromatic number* $\chi''(G)$ of G is the smallest integer k such that G has a total k-coloring. For a total k-coloring ϕ of G, we use $C_\phi(v) = \{\phi(v)\} \cup \{\phi(xv)|xv \in E(G)\}$ to denote the set of colors assigned to a vertex v and those edges incident with v. The total k-coloring ϕ is called *neighbor-distinguishing* (or ϕ is an AVDT-k-coloring) if $C_\phi(u) \neq C_\phi(v)$ for each edge $uv \in E(G)$. The *neighbor-distinguishing total chromatic number* $\chi''_a(G)$ of G is the smallest integer k such that G has an AVDT-k-coloring.

It holds trivially that $\chi''_a(G) \geq \chi''(G) \geq \Delta + 1$ for any graph G. Moreover, if G contains two adjacent vertices of maximum degree, then it is evident that $\chi''_a(G) \geq \Delta + 2$. Zhang et al. [16] introduced the neighbor-distinguishing total coloring of graphs and proposed the following conjecture:

Conjecture 1. *If G is a graph with $|V(G)| \geq 2$, then $\chi''_a(G) \leq \Delta + 3$.*

Conjecture 1 was confirmed for graphs with $\Delta = 3$ in [2,8,10,12], and graphs with $\Delta = 4$ in [9,11]. Applying a probabilistic analysis, Coker and Johannson [4] established an upper bound $\Delta + C$ for $\chi''_a(G)$, where C is a positive constant. Huang, Wang and Yan [7] proved that $\chi''_a(G) \leq 2\Delta$ for any graph G with $\Delta \geq 3$.

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Suppose that G is a planar graph. Huang and Wang [6] showed that if $\Delta \geq 11$, then G satisfies Conjecture 1. This result was improved to the case $\Delta = 10$ by Cheng et al. [3], and furthermore to the case $\Delta = 9$ by Wang et al. [14] and Hu et al. [5], independently. Therefore the following theorem holds:

Theorem 1. *If G is a planar graph with $\Delta \geq 9$, then $\chi''_a(G) \leq \Delta + 3$.*

In 2014, Wang and Huang [13] showed: (i) if G is a planar graph with $\Delta \geq 13$, then $\chi''_a(G) \leq \Delta + 2$; and (ii) if moreover $\Delta \geq 14$, then $\chi''_a(G) = \Delta + 2$ if and only if G contains two adjacent vertices of maximum degree. Recently, the conclusion (i) was extended to the case $11 \leq \Delta \leq 12$ by Yang et al. [15]. Thus, combining these facts, we obtain the following theorem:

Theorem 2. *If G is a planar graph with $\Delta \geq 11$, then $\chi''_a(G) \leq \Delta + 2$.*

The purpose of this paper is to extend the above result (ii), namely we will show that every planar graph G with $\Delta = 13$ has $\chi''_a(G) = 15$ if and only if G contains two adjacent vertices of degree 13.

2. Preliminaries

Suppose that H is a subgraph of a plane graph G. For $x \in V(H) \cup F(H)$, let $d_H(x)$ denote the degree of x in H. A vertex of degree k (at least k, at most k) in H is called a *k-vertex* (*k^+-vertex, k^--vertex*). Similarly, we can define k-face, k^+-face and k^--face. For a vertex $v \in V(H)$, let $N_H(v)$ denote the set of neighbors of v in H. A *k-neighbor* of v is a k-vertex adjacent to v. Let $N^H_k(v)$ denote the set of k-neighbors of v in H, and set $d^H_k(v) = |N^H_k(v)|$. Similarly, we can define $d^H_{k^+}(v)$ and $d^H_{k^-}(v)$.

A 4-cycle is said to be *bad* if it has at least one 2-vertex, and *special* if it has two non-adjacent 2-vertices. A 3-face is *special* if it is incident to a 2-vertex. A 2-vertex is called *special* if it is adjacent to a special 4-cycle. A k-vertex v, with $k \geq 3$, is *bad* if each of the faces incident to it is either a 3-face, or a 4-face whose boundary forms a bad 4-cycle. We use $d^H_{kb}(v)$ to denote the number of bad k-vertices adjacent to v. If there is no confusion in the context, we omit the letter G in $d_G(v)$, $d^G_k(v)$, $d^G_{k^+}(v)$, $d^G_{k^-}(v)$ and $d^G_{kb}(v)$.

Suppose that ϕ is an AVDT-k-coloring of a graph G and $v \in V(G)$. Let $m_\phi(v)$ denote the sum of colors in $C_\phi(v)$. Obviously, for two adjacent vertices u and v, if $m_\phi(u) \neq m_\phi(v)$, then $C_\phi(u) \neq C_\phi(v)$. Two adjacent vertices u and v are called *conflict* under ϕ if $C_\phi(u) = C_\phi(v)$. An edge uv is said to be *legally colored* if its color is different from that of its adjacent elements in $V(G) \cup E(G)$ and no pair of new conflict vertices are produced.

The following famous Combinatorial Nullstellensatz will be frequently used in our proof.

Lemma 1 (*Combinatorial Nullstellensatz, [1]*)**.** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Assume that the degree $\deg(P)$ of P equals $\sum_{i=1}^{n} k_i$ and the coefficient of $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in P is non-zero, where each k_i is a non-negative integer. If $S_1, S_2, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| > k_i$, then there are $s_i \in S_i$ for $i = 1, 2, \ldots, n$ so that $P(s_1, s_2, \ldots, s_n) \neq 0$.*

3. Main results

The main result of this paper is as follows, whose proof is based on meticulous structural analysis and powerful discharging technique.

Theorem 3. *Let G be a planar graph with $\Delta = 13$. Then $\chi''_a(G) = 15$ if and only if G contains two adjacent 13-vertices.*

Proof. If G contains two adjacent 13-vertices, then the previous discussion claims that $\chi''_a(G) \geq \Delta + 2 = 15$. Conversely, assume that G does not contain adjacent 13-vertices. It suffices to show $\chi''_a(G) \leq 14$. Suppose that this is not true. Let G be a counterexample with $|V(G)| + |E(G)|$ being as small as possible. Obviously, G is connected. For any edge $e \in E(G)$, let $H = G - e$. It is easy to see that $12 \leq \Delta(H) \leq 13$. If $\Delta(H) = 12$, then $\chi''_a(H) \leq 12 + 2 = 14$ by Theorem 2. If $\Delta(H) = 13$, then $\chi''_a(H) \leq 14$ by the minimality of G. Hence we always have $\chi''_a(H) \leq 14$. Such discussion will be omitted in the following proof. Since G contains no adjacent 13-vertices, it is easy to verify that no leaf is adjacent to a 13-vertex. Let $C = [1, 14]$ denote a set of 14 colors.

Remark 1. Assume that $v \in V(G)$ is a k-vertex with neighbors $v_1, v_2, \ldots, v_k$, where $1 \leq k \leq 6$. Let ϕ be a partial AVDT-14-coloring of G with v uncolored. Assume that $\phi(vv_i) = i$ for $i \in [1, k]$. Set $|\{\phi(v_1), \phi(v_2), \ldots, \phi(v_k)\} \cap [1, k]| = p$, say $\phi(v_1), \phi(v_2), \ldots, \phi(v_p) \in [1, k]$. Since $14 - k - (k - p) = (14 - 2k) + p \geq p + 2$, we can color v with a color in $[k+1, 14] \setminus \{\phi(v_{p+1}), \phi(v_{p+2}), \ldots, \phi(v_k)\}$ such that v does not conflict with $v_1, v_2, \ldots, v_p$. Hence ϕ is extended to the whole graph G.

By virtue of Remark 1, to obtain an AVDT-14-coloring of G, we may first erase the colors of 6^--vertices and finally recolor them after other vertices and edges have been legally colored.

Claim 1. *There is no edge $uv \in E(G)$ such that $d(v) \leq 7$ and $d(u) \leq 5$.*