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On a problem of partitions of the set of nonnegative integers with the same representation functions

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ABSTRACT

Dombi has shown that the set $\mathbb{N}$ of all non-negative integers can be partitioned into two subsets with identical representation functions. In this paper, we prove that one cannot partition $\mathbb{N}$ into more than two subsets with identical representation functions, while for any integer $k \geq 3$ there is a partition $\mathbb{N} = A_1 \cup \cdots \cup A_k$ such that A_i and A_{k+1-i} have the same representation function for any integer $1 \leq i \leq k$.

1. Introduction

Let $\mathbb{N}$ be the set of all nonnegative integers. For a given set $S \subset \mathbb{N}$ the representation function $R_S(n)$ is defined as the number of solutions of the equation $n = s + s', s < s', s, s' \in S$. In 2002, using the additive properties of Thue–Morse sequence, Dombi [3] constructed two sets of nonnegative integers with infinite symmetric difference such that the corresponding representation functions are identical.

Theorem A (*See [3], Theorem 1*)**.** *The set of positive integers can be partitioned into two subsets A and B such that $R_A(n) = R_B(n)$ for every nonnegative integer n.*

Let $\mathcal{A}$ be the set of those nonnegative integers with an even sum of binary digits in their base-2 representation, and let $\mathcal{B} = \mathbb{N} \setminus \mathcal{A}$. The following extension of Dombi's result, implicitly present in [6], has appeared in an explicit form in [5].

Theorem B (*See [5], Theorem 2*)**.** *Let C and D be sets of nonnegative integers such that $C \cup D = \mathbb{N}$ and $C \cap D = \emptyset$, $0 \in C$. Then $R_C(n) = R_D(n)$ for every nonnegative integer n if and only if $C = \mathcal{A}$ and $D = \mathcal{B}$.*

Dombi's paper was followed by a series of other papers studying a number of related problems; see, for instance, [1,2,4,7,8,10]. In [9], the first present author posed the following problem:

Problem 1. Given a positive integer $k (k \geq 3)$, does there exist a partition

$$\mathbb{N} = \bigcup_{m=1}^{k} A_m, \quad A_u \cap A_v = \emptyset, \ u \neq v$$

such that $R_{A_u}(n) = R_{A_v}(n)$ $(1 \leq u \neq v \leq k)$ for all sufficiently large integers n?

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In this paper, we focus on Problem 1 and obtain the following results:

Theorem 1.1. *Given a positive integer $k \geq 3$, there is no partition*

$$\mathbb{N} = \bigcup_{j=1}^{k} A_j, \; A_u \cap A_v = \emptyset, \; 1 \leq u \neq v \leq k$$

such that $R_{A_u}(n) = R_{A_v}(n)$ for every nonnegative integer n.

Theorem 1.2. *Given a positive integer $k \geq 2$, there exists a partition*

$$\mathbb{N} = \bigcup_{j=1}^{k} A_j, \; A_u \cap A_v = \emptyset, \; 1 \leq u \neq v \leq k$$

such that for $1 \leq i \leq k$ we have $R_{A_i}(n) = R_{A_{k+1-i}}(n)$ for every nonnegative integer n.

2. Proof of Theorem 1.1

Assume that there exists a partition

$$\mathbb{N} = \bigcup_{j=1}^{k} A_j, \; A_u \cap A_v = \emptyset, \; 1 \leq u \neq v \leq k$$

such that for every nonnegative integer n. Then we have

$$R_{A_u}(n) = R_{A_v}(n). \tag{2.1}$$

Without loss of generality, we may assume that

$$\min A_1 < \min A_2 < \cdots < \min A_k. \tag{2.2}$$

Fact I. For $j = 1, \ldots, k$, we have $|A_j \cap [0, k-1]| = 1$.

In fact, if there exists an integer $u \in \{1, \ldots, k\}$ such that

$$|A_u \cap [0, k-1]| \geq 2, \tag{2.3}$$

then there exists an integer $v \in \{1, \ldots, k\}$ such that

$$A_v \cap [0, k-1] = \emptyset. \tag{2.4}$$

By (2.3), we may assume that there exist two integers $0 \leq h \neq t \leq k-1$ and $h, t \in A_u$ such that

$$R_{A_u}(h+t) \geq 1.$$

By (2.4) we know that if $R_{A_v}(n) \geq 1$ for $0 \leq n \leq 3k-1$, then $n \geq 2k$. Noting that $0 < h+t < 2k-2$, we have

$$R_{A_v}(h+t) = 0.$$

Thus $R_{A_u}(h+t) \neq R_{A_v}(h+t)$, which contradicts (2.1).

By Fact I and (2.2) we have

$$j-1 \in A_j \cap [0, k-1], \quad j = 1, \ldots, k. \tag{2.5}$$

Fact II. For $j = 1, \ldots, k$, we have

$$A_j \cap [k, 3k-1] = \{2k-j, 3k-j\}. \tag{2.6}$$

In fact, if there exists an integer $u' \in \{1, \ldots, k\}$ such that $|A_{u'} \cap [k, 2k-1]| \geq 2$, then there exists an integer $v' \in \{1, \ldots, k\}$ such that

$$A_{v'} \cap [k, 2k-1] = \emptyset. \tag{2.7}$$

Assume that $k \in A_p$ for some $p \in \{1, \ldots, k\}$. By (2.5), we have

$$R_{A_p}(k+p-1) \geq 1.$$