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Chromatic bounds for some classes of $2K_2$-free graphs

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ABSTRACT

A hereditary class $\mathcal{G}$ of graphs is χ-*bounded* if there is a χ-*binding function*, say f such that $\chi(G) \leq f(\omega(G))$, for every $G \in \mathcal{G}$, where $\chi(G)$ ($\omega(G)$) denotes the chromatic (clique) number of G. It is known that for every $2K_2$-free graph G, $\chi(G) \leq \binom{\omega(G)+1}{2}$, and the class of $(2K_2, 3K_1)$-free graphs does not admit a linear χ-binding function. In this paper, we are interested in classes of $2K_2$-free graphs that admit a linear χ-binding function. We show that the class of $(2K_2, H)$-free graphs, where $H \in \{K_1 + P_4, K_1 + C_4, \overline{P_2 \cup P_3}, HVN, K_5 - e, K_5\}$ admits a linear χ-binding function. Also, we show that some superclasses of $2K_2$-free graphs are χ-bounded.

1. Introduction

All our graphs in this paper are simple, finite and undirected. For notation and terminology that are not defined here, we refer to West [21]. As usual, let P_n, C_n, K_n denote the induced path, induced cycle and complete graph on n vertices respectively. Let $K_{p,q}$ be the complete bipartite graph with classes of size p and q. For a graph G, the complement of G is denoted by $\overline{G}$. If $\mathcal{F}$ is a family of graphs, a graph G is said to be $\mathcal{F}$-*free* if it contains no induced subgraph isomorphic to any member of $\mathcal{F}$. If G_1 and G_2 are two vertex disjoint graphs, then their *union* $G_1 \cup G_2$ is the graph with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$. Similarly, their *join* $G_1 + G_2$ is the graph with $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{(x, y) \mid x \in V(G_1),\ y \in V(G_2)\}$. For any positive integer k, kG denotes the union of k graphs each isomorphic to G.

A *proper coloring* (or simply *coloring*) of a graph G is an assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors required to color G is called the *chromatic number* of G, and is denoted by $\chi(G)$. A *clique* in a graph G is a set of vertices that are pairwise adjacent in G. The *clique number* of G, denoted by $\omega(G)$, is the size of a maximum clique in G. Obviously, for any graph G, we have $\chi(G) \geq \omega(G)$. The existence of triangle-free graphs with large chromatic number (see [17] for a construction of such graphs) shows that for a general class of graphs, there is no upper bound on the chromatic number as a function of clique number.

A graph G is called *perfect* if $\chi(H) = \omega(H)$, for every induced subgraph H of G; otherwise it is called *imperfect*. A hereditary class $\mathcal{G}$ of graphs is said to be χ-*bounded* [11] if there exists a function f (called a χ-*binding function* of $\mathcal{G}$) such that $\chi(G) \leq f(\omega(G))$, for every $G \in \mathcal{G}$. If $\mathcal{G}$ is the class of H-free graphs for some graph H, then f is denoted by f_H. We refer to [18] for an extensive survey of χ-bounds for various classes of graphs.

The class of $2K_2$-free graphs and its related classes have been well studied in various contexts in the literature; see [3]. Here, we would like to focus on showing χ-binding functions for some classes of graphs related to $2K_2$-free graphs. Wagon [20] showed that the class of mK_2-free graphs admits an $O(x^{2m-2})$ χ-binding function for all $m \geq 1$. In particular, he

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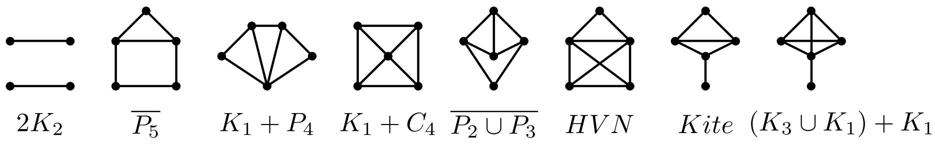

Fig. 1. Some special graphs.

Table 1
Known chromatic bounds for $(2K_2, H)$-free graphs, where H is any $2K_2$-free graph on 5 vertices with $\alpha(H) = 2$, and the graph $X \in \{Kite, K_4 \cup K_1, (K_3 \cup K_1) + K_1\}$.

Graph class $\mathcal{C}$	χ-bound for $G \in \mathcal{C}$	
$(2K_2, \overline{P_5})$-free graphs	$\lfloor \frac{3\omega(G)}{2} \rfloor$	[10]
$(2K_2, C_5)$-free graphs	$\omega(G)^{3/2}$	[12]
$(2K_2, K_1 + P_4)$-free graphs	$\omega(G) + 1$	(Corollary 1)
$(2K_2, K_1 + C_4)$-free graphs	$\omega(G) + 5$	(Corollary 2)
$(2K_2, \overline{P_2 \cup P_3})$-free graphs	$\omega(G) + 1$	(Corollary 3)
$(2K_2, HVN)$-free graphs	$\omega(G) + 3$	(Corollary 4)
$(2K_2, K_5 - e)$-free graphs	$\omega(G) + 4$	(Corollary 5)
$(2K_2, K_5)$-free graphs	$2\omega(G) + 1 \leq 9$	(Corollary 6)
$(2K_2, X)$-free graphs	$\binom{\omega(G)+1}{2}$	[20]

showed that $f_{2K_2}(x) = \binom{x+1}{2}$, and the best known lower bound is $\frac{R(C_4, K_{x+1})}{3}$, where $R(C_4, K_{x+1})$ denotes the smallest k such that every graph on k vertices contains either a clique of size $x + 1$ or $2K_2$ [11]. This lower bound is non-linear because Chung [8] showed that $R(C_4, K_t)$ is at least $t^{1+\epsilon}$ for some $\epsilon > 0$. It is interesting to note that Brause et al. [4] showed that the class of $(2K_2, 3K_1)$-free graphs does not admit a linear χ-binding function. It follows that the class of $(2K_2, H)$-free graphs, where H is any $2K_2$-free graph with independence number $\alpha(H) \geq 3$, does not admit a linear χ-binding function.

Here we are interested in classes of $2K_2$-free graphs that admit a linear χ-binding function, in particular, some classes of $2K_2$-free graphs that admit a 'special' linear χ-binding function $f(x) = x + c$, where c is a positive integer, that is, $2K_2$-free graphs G such that $\chi(G) \leq \omega(G) + c$. If $c = 1$, then this special upper bound is called the *Vizing bound* for the chromatic number, and is well studied in the literature; see [14,18] and the references therein. Brause et al. [4] showed that if G is a connected $(2K_2, K_{1,3})$-free graph with independence number $\alpha(G) \geq 3$, then G is perfect. It follows from a result of [13] that if G is a $(2K_2, \text{paw})$-free graph, then either G is perfect or $\chi(G) = 3$ and $\omega(G) = 2$ (see also [4]). Nagy and Szentmiklóssy (see [11]) showed that if G is a $(2K_2, K_4)$-free graph, then $\chi(G) \leq 4$. Blaszik et al. [2] and independently Gyárfás [11] showed that if G is a $(2K_2, C_4)$-free graph, then $\chi(G) \leq \omega(G) + 1$, and the equality holds if and only if G is not a split-graph. It follows from a result of [14] that if G is a $(2K_2, K_4 - e)$-free graph, then $\chi(G) \leq \omega(G) + 1$. Fouquet et al. [10] showed that if G is a $(2K_2, \overline{P_5})$-free graph, then $\chi(G) \leq \left\lfloor \frac{3\omega(G)}{2} \right\rfloor$, and the bound is tight. Brause et al. [4] showed that if G is a $(2K_2, K_1 + P_4)$-free graph, then $\chi(G) \leq 2\omega(G)$.

In this paper, by using structural results, we show that the class of $(2K_2, H)$-free graphs, where $H \in \{K_1 + P_4, K_1 + C_4, \overline{P_2 \cup P_3}, HVN, K_5 - e\}$ admits a special linear χ-binding function $f(x) = x + c$, where c is a positive integer; see Fig. 1. We also show that the class of $(2K_2, K_5)$-free graphs admits a linear χ-binding function. Table 1 shows the known chromatic bounds for a $(2K_2, H)$-free graph G, where H is any $2K_2$-free graph on 5 vertices with $\alpha(H) = 2$. We remark that some of the cited bounds are consequences of much stronger results available in the literature. Finally, we show χ-binding functions for some superclasses of $2K_2$-free graphs.

2. Notation, terminology, and preliminaries

For a positive integer k, we simply write $\langle k \rangle$ to denote the set $\{1, 2, \ldots, k\}$.

Let G be a graph, with vertex-set $V(G)$ and edge-set $E(G)$. For $x \in V(G)$, $N(x)$ denotes the set of all neighbors of x in G. For any two disjoint subsets $S,\ T \subseteq V(G)$, $[S, T]$ denotes the set of edges $\{e \in E(G) \mid e$ has one end in S and the other in $T\}$. Also, for $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S in G, and for convenience we simply write $[S]$ instead of $G[S]$. If H is an induced subgraph of G, then we say that G contains H. Note that if H_1 and H_2 are any two graphs, and if G is (H_1, H_2)-free, then $\overline{G}$ is $(\overline{H_1}, \overline{H_2})$-free.

A *k-clique covering* of a graph G is a partition $(V_1, V_2, \ldots, V_k)$ of $V(G)$ such that V_i is a clique, for each $i \in \langle k \rangle$. The *clique covering number* of the graph G, denoted by $\theta(G)$, is the minimum integer k such that G admits a k-clique covering. An *independent/stable* set in a graph G is a set of vertices that are pairwise non-adjacent in G. The *independence number* of G, denoted by $\alpha(G)$, is the size of a maximum independent set in G. Clearly, for any graph G, we have $\chi(G) = \theta(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$.