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# Chromatic numbers of spheres

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ABSTRACT

The chromatic number of a subset of Euclidean space is the minimal number of colors sufficient for coloring all points of this subset in such a way that any two points at the distance 1 have different colors. We give new upper bounds on chromatic numbers of spheres. This also allows us to give new upper bounds on chromatic numbers of any bounded subsets.



## 1. Introduction

The *chromatic number* of the Euclidean space $\chi(\mathbb{R}^n)$ is the minimal number of colors sufficient for coloring all points of $\mathbb{R}^n$ in such a way that any two points at the distance 1 have different colors. Determining $\chi(\mathbb{R}^n)$ is considered as an important problem of discrete geometry. For the history and the overview of this problem see [30,33–35,41].

Even in the case $n = 2$ the exact value of the chromatic number is unknown. The best current bounds are

$$5 \leqslant \chi(\mathbb{R}^2) \leqslant 7,$$

The lower bound is due to de Gray [12], the upper is due to Isbell [41].

In the case of arbitrary $n$ Raigorodskii (the lower bound, [29]), Larman and Rogers (the upper bound, [20]) proved that

$$(1.239 + o(1))^n \leqslant \chi(\mathbb{R}^n) \leqslant (3 + o(1))^n.$$

For small values of $n$ better lower bounds are known. For the overview of the recent progress we refer to [10,11].

The chromatic number may be defined for an arbitrary metric space (see for example [18,19]). It also can be defined with any positive real number instead of 1 in the definition of the chromatic number (we call this number the *forbidden distance*). The space $\mathbb{R}^n$ admits homothety, therefore, $\chi(\mathbb{R}^n)$ does not depend on the choice of the forbidden distance, but in general case the chromatic number essentially depends on it.

We consider the case of a spherical space. Let $\chi(S^n_R)$ be the minimal number of colors needed to color the Euclidean sphere $S^n_R$ of radius $R$ in $\mathbb{R}^{n+1}$ in such a way that any two points of $S^n_R$ at the Euclidean distance 1 have different colors. It is clear that $\chi(S^n_{1/2}) = 2$ and $\chi(S^n_R) = 1$ for $R < 1/2$. Note that this is the case, when the chromatic number depends on the forbidden distance. But the problem of determining the chromatic number of $S^n_R$ with a forbidden distance $d$ is equivalent to determining the chromatic number of $S^n_{R/d}$ with the forbidden distance 1.

In 1981 Erdős conjectured that for any fixed $R > 1/2$, $\chi(S^n_R)$ is growing as $n$ tends to infinity. In 1983 this was proved by Lovász [22] using an interesting mixture of combinatorial and topological techniques. Among other things, in this paper

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Lovász claimed that for $R < \sqrt{\frac{n+1}{2n+4}}$ (i.e. when the side of regular $(n+1)$-dimensional simplex inscribed in our sphere is less than 1) we have $\chi(S_R^n) = n+1$. However, in 2012 Raigorodskii [31,32] showed that this statement is wrong. In [32] it was shown that actually for any fixed $R > 1/2$ the quantity $\chi(S_R^n)$ is growing exponentially. Some improvements of lower bounds were obtained in [16,17].

It is clear that

$$\chi(S_R^n) \leqslant (3 + o(1))^n$$

because $S_R^n$ is a subset of $\mathbb{R}^{n+1}$. Despite the remarkable interest to this problem there are no better upper bounds in general. For spheres of small radii ($R < 3/2$) the work of Rogers [37] easily implies a much stronger bound. Consider a spherical cap on $S_R^n$ of such radius that the Euclidean diameter of this cap is less than 1. Then we cover $S_R^n$ with copies of this cap and paint every cap in its own color. This establishes the bound

$$\chi(S_R^n) \leqslant (2R + o(1))^n.$$

In this paper we prove a new upper bound on $\chi(S_R^n)$ in the case of $R > \frac{\sqrt{5}}{2}$. More precisely, define

$$x(R) = \begin{cases} \sqrt{5 - \dfrac{2}{R^2} + 4\sqrt{1 - \dfrac{5R^2-1}{4R^4}}}, & R > \dfrac{\sqrt{5}}{2} \\ 2R, & \dfrac{1}{2} < R \leqslant \dfrac{\sqrt{5}}{2} \end{cases}$$

**Theorem 1.** *For $R > \frac{1}{2}$ we have $\chi(S_R^n) \leqslant (x(R) + o(1))^n$.*

It is clear that the base of exponent is always less than 3. (However, it tends to 3 as $R$ tends to infinity.) Further, it will be evident that it is less than $2R$ over the interval $\left(\frac{\sqrt{5}}{2}; \frac{3}{2}\right)$. Thus, we improve the current bounds for all $R$ that is not in the interval $\frac{1}{2} < R \leqslant \frac{\sqrt{5}}{2}$. In the latter case the method of our proof breaks down, but it provides another proof of the bound $\chi(S_R^n) \leqslant (2R + o(1))^n$.

Let $B_R^{n+1} \subset \mathbb{R}^{n+1}$ be a Euclidean ball of radius $R$ (centered in the origin). By $\chi(B_R^{n+1})$ denote the chromatic number of $B^{n+1}$ (with forbidden distance 1). The construction in the proof of Theorem 1 also implies the following

**Theorem 2.** *For $R > \frac{1}{2}$ we have $\chi(B_R^{n+1}) \leqslant (x(R) + o(1))^n$.*

The Erdős–de Bruijn theorem [9] states that the chromatic number of the Euclidean space $\mathbb{R}^n$ is reached at some finite distance graph embedded in this space. Hence, Theorem 2 connects $\chi(\mathbb{R}^n)$ with the radius of circumscribed sphere of this graph. The author is grateful to A.B. Kupavskii for the remark that Theorem 1 should imply Theorem 2.

It is of interest to mention the problem of determining *the measurable chromatic number* $\chi_m(\mathbb{R}^n)$, which is defined in the same way, but with the extra condition that all monocolored sets are required to be measurable. In this case upper bounds remain to be the same, but additional analytic techniques can be applied to establish better lower bounds. Thus, in [3] in was proved that

$$(1.268 - o(1))^n \leqslant \chi_m(\mathbb{R}^n).$$

The best lower bound for $n = 2$ was obtained by Falconer [14], the best lower bounds for some other small values of $n$ can be found in [13].

Another fruitful area of research is to consider colorings with more restrictions on a monocolored set. Some results in this direction were obtained in [4–6,26,28,38–40].

This paper is organized as follows. The Section 2 contains a brief exposition of our technique and provides further references to the bibliography. In Section 3 we set up notation and terminology. Section 4 is devoted to the proof of Theorem 1 and Section 5 provides the implication of Theorem 2.

## 2. Summary of the technique

Generally, in our proof of Theorem 1 we follow the approach of Larman and Rogers in [20] about the chromatic number of the Euclidean space. We construct some set on $S_R^n$ without a pair of points at the distance 1, bound its density and then cover the whole sphere with copies of this set. But the realization of every item of this plan in [20] cannot be generalized directly to the spherical case. For instance, the construction of the set without distance 1 is strongly based on theory of lattices in $\mathbb{R}^n$. Therefore, we should provide some new ideas for our case.

During our proof we should turn to geometric covering problems. We need to cover a sphere with copies of some disconnected set. There is a well-developed theory of economical coverings. The most common tool is the Rogers theorem [36] (and its relatives) on periodical coverings of a Euclidean space with copies of a convex body. There are different approaches to prove results of that type. Most of proofs essentially use the convexity of the body and special properties of Euclidean spaces. This is not appropriate in our situation.