Discrete Mathematics 341 (2018) 3148–3151

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

A note on the DP-chromatic number of complete bipartite graphs

Jeffrey A. Mudrock

Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, United States

ARTICLE INFO

Article history:
Received 24 March 2018
Received in revised form 1 August 2018
Accepted 2 August 2018

Keywords:
Graph coloring
List coloring
DP-coloring

ABSTRACT

DP-coloring (also called correspondence coloring) is a generalization of list coloring recently introduced by Dvořák and Postle. Several known bounds for the list chromatic number of a graph G, $\chi_\ell(G)$, also hold for the DP-chromatic number of G, $\chi_{DP}(G)$. On the other hand, there are several properties of the DP-chromatic number that show that it differs with the list chromatic number. In this note we show one such property. It is well known that $\chi_\ell(K_{k,t}) = k + 1$ if and only if $t \geq k^k$. We show that $\chi_{DP}(K_{k,t}) = k + 1$ if $t \geq 1 + (k^k/k!)(\log(k!) + 1)$, and we show that $\chi_{DP}(K_{k,t}) < k + 1$ if $t < k^k/k!$.

1. Introduction

In this note all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [13] for terminology and notation. For this note the set of natural numbers is $\mathbb{N} = \{1, 2, 3 \ldots\}$. The natural log function is denoted log. Given a set A, $\mathcal{P}(A)$ is the power set of A. Also, for any $k \in \mathbb{N}$, $[k] = \{1, 2, 3, \ldots, k\}$. If G is a graph and $S, U \subseteq V(G)$, we use $G[S]$ for the subgraph of G induced by S, and we use $E_G(S, U)$ for the subset of $E(G)$ with one endpoint in S and one endpoint in U. Also, if $v \in V(G)$ we use $N_G(v)$ for the set of neighbors of v in G.

1.1. List coloring

List coloring is a well known variation on the classic vertex coloring problem, and it was introduced independently by Vizing [12] and Erdős, Rubin, and Taylor [7] in the 1970s. In the classic vertex coloring problem we wish to color the vertices of a graph G with as few colors as possible so that adjacent vertices receive different colors, a so-called *proper coloring*. The chromatic number of a graph, denoted $\chi(G)$, is the smallest k such that G has a proper coloring that uses k colors. For list coloring, we associate a *list assignment*, L, with a graph G such that each vertex $v \in V(G)$ is assigned a list of colors $L(v)$ (we say L is a list assignment for G). The graph G is *L-colorable* if there exists a proper coloring f of G such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to f as a *proper L-coloring* of G). A list assignment L is called a *k-assignment* for G if $|L(v)| = k$ for each $v \in V(G)$. The *list chromatic number* of a graph G, denoted $\chi_\ell(G)$, is the smallest k such that G is L-colorable whenever L is a k-assignment for G. We say G is *k-choosable* if $k \geq \chi_\ell(G)$.

It is immediately obvious that for any graph G, $\chi(G) \leq \chi_\ell(G)$. Erdős, Rubin, and Taylor [7] studied the choosability of $K_{m,m}$ and observed that if $m = \binom{2k-1}{k}$, then $\chi_\ell(K_{m,m}) > k$. The following related result is often attributed to Vizing [12] or Erdős, Rubin, and Taylor [7], but it is best described as a folklore result.

Theorem 1. *For $k \in \mathbb{N}$, $\chi_\ell(K_{k,t}) = k + 1$ if and only if $t \geq k^k$.*

We study the analogue of Theorem 1 for DP-coloring.

E-mail address: jmudrock@hawk.iit.edu.

https://doi.org/10.1016/j.disc.2018.08.003
0012-365X/

1.2. DP-coloring

Dvořák and Postle [6] introduced DP-coloring (they called it correspondence coloring) in 2015 in order to prove that every planar graph without cycles of lengths 4 to 8 is 3-choosable. Intuitively, DP-coloring is a generalization of list coloring where each vertex in the graph still gets a list of colors but identification of which colors are different can vary from edge to edge. Following [5], we now give the formal definition. Suppose G is a graph. A *cover* of G is a pair $\mathcal{H} = (L, H)$ consisting of a graph H and a function $L : V(G) \to \mathcal{P}(V(H))$ satisfying the following four requirements:

(1) the sets $\{L(u) : u \in V(G)\}$ form a partition of $V(H)$;
(2) for every $u \in V(G)$, the graph $H[L(u)]$ is complete;
(3) if $E_H(L(u), L(v))$ is nonempty, then $u = v$ or $uv \in E(G)$;
(4) if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

Suppose $\mathcal{H} = (L, H)$ is a cover of G. We say $\mathcal{H}$ is *k-fold* if $|L(u)| = k$ for each $u \in V(G)$. An $\mathcal{H}$-coloring of G is an independent set in H of size $|V(G)|$. It is immediately clear that an independent set $I \subseteq V(H)$ is an $\mathcal{H}$-coloring of G if and only if $|I \cap L(u)| = 1$ for each $u \in V(G)$.

The *DP-chromatic number* of a graph G, $\chi_{DP}(G)$, is the smallest $k \in \mathbb{N}$ such that G admits an $\mathcal{H}$-coloring for every k-fold cover $\mathcal{H}$ of G. Suppose we wish to prove $\chi_{DP}(G) \leq k$. Since every k-fold cover of G is isomorphic to a subgraph of some k-fold cover, $\mathcal{H}' = (L', H')$, of G with the property that $E_{H'}(L'(u), L'(v))$ is a perfect matching whenever $uv \in E(G)$, we need only show that G has an $\mathcal{H}$-coloring whenever $\mathcal{H} = (L, H)$ is a k-fold cover of G such that $E_H(L(u), L(v))$ is a perfect matching for each $uv \in E(G)$.

Given a list assignment, L, for a graph G, it is easy to construct a cover $\mathcal{H}$ of G such that G has an $\mathcal{H}$-coloring if and only if G has a proper L-coloring (see [5]). It follows that $\chi_\ell(G) \leq \chi_{DP}(G)$. This inequality may be strict since it is easy to prove that $\chi_{DP}(C_n) = 3$ whenever $n \geq 3$, but the list chromatic number of any even cycle is 2 (see [5] and [7]).

We now briefly discuss some similarities between DP-coloring and list coloring. First, notice that like k-choosability, the graph property of having DP-chromatic number at most k is monotone. It is also clear that, as in the context of list coloring, if $\chi_{DP}(G) = k$, then an $\mathcal{H}$-coloring of G exists whenever $\mathcal{H}$ is an m-fold cover of G with $m \geq k$. The *coloring number* of a graph G, denoted col(G), is the smallest integer d for which there exists an ordering, $v_1, v_2, \dots, v_n$, of the elements in $V(G)$ such that each vertex v_i has at most $d-1$ neighbors among $v_1, v_2, \dots, v_{i-1}$. It is easy to prove that $\chi_\ell(G) \leq \chi_{DP}(G) \leq \text{col}(G)$. Thomassen [11] famously proved that every planar graph is 5-choosable, and Dvořák and Postle [6] observed that the DP-chromatic number of every planar graph is at most 5. Also, Molloy [10] recently improved a theorem of Johansson [9] by showing that every triangle-free graph G with maximum degree $\Delta(G)$ satisfies $\chi_\ell(G) \leq (1 + o(1))\Delta(G)/\log(\Delta(G))$. Bernshteyn [3] subsequently showed that this bound also holds for the DP-chromatic number.

On the other hand, Bernshteyn [4] showed that if the average degree of a graph G is d, then $\chi_{DP}(G) = \Omega(d/\log(d))$. This is in stark contrast to the celebrated result of Alon [1] which says $\chi_\ell(G) = \Omega(\log(d))$. It was also recently shown in [5] that there exist planar bipartite graphs with DP-chromatic number 4 even though the list chromatic number of any planar bipartite graph is at most 3 [2]. A famous result of Galvin [8] says that if G is a bipartite multigraph and $L(G)$ is the line graph of G, then $\chi_\ell(L(G)) = \chi(L(G)) = \Delta(G)$. However, it is also shown in [5] that every d-regular graph G satisfies $\chi_{DP}(L(G)) \geq d + 1$.

1.3. Outline of results and an open question

In this note we present some results on the DP-chromatic number of complete bipartite graphs. By what was mentioned in the previous subsection, we know that if $k, t \in \mathbb{N}$, $\chi_{DP}(K_{k,t}) \leq \text{col}(K_{k,t}) \leq k + 1$. For the remainder of this note, for each $k \in \mathbb{N}$, let $\mu(k)$ be the smallest natural number l such that $\chi_{DP}(K_{k,l}) = k + 1$. We have that $\mu(k)$ exists for each $k \in \mathbb{N}$ since we know by Theorem 1,

$$k + 1 = \chi_\ell(K_{k,k^k}) \leq \chi_{DP}(K_{k,k^k}) \leq k + 1.$$

This means that $\mu(k) \leq k^k$ for each $k \in \mathbb{N}$. The following proposition is also clear.

Proposition 2. *For $k \in \mathbb{N}$, $\chi_{DP}(K_{k,t}) = k + 1$ if and only if $t \geq \mu(k)$*

Proof. If $t \geq \mu(k)$, then $k + 1 = \chi_{DP}(K_{k,\mu(k)}) \leq \chi_{DP}(K_{k,t}) \leq k + 1$ since $K_{k,\mu(k)}$ is a subgraph of $K_{k,t}$. Conversely, if $\chi_{DP}(K_{k,t}) = k + 1$, then $\mu(k) \leq t$ by the definition of $\mu(k)$. □

Computing $\mu(k)$ is easy when $k = 1,2$. Clearly, $\mu(1) = 1$. Also, $\mu(2) = 2$ follows from the fact that $\chi_{DP}(K_{2,1}) \leq \text{col}(K_{2,1}) = 2$ and the fact that $K_{2,2}$ is a 4-cycle which implies $\chi_{DP}(K_{2,2}) = 3$. We have a tedious argument that shows $\mu(3) = 6$, which for the sake of brevity, we do not present in this note. The following question leads to the discovery of both results in this note.

Question 3. *For each $k \geq 4$, what is the exact value of $\mu(k)$?*

We obtain an upper bound and lower bound on $\mu(k)$. Our first result gives us a lower bound.