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On two upper bounds for hypersurfaces involving a Thas' invariant

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Let X^n be a hypersurface in $\mathbb{P}^{n+1}$ with $n \geq 1$ defined over a finite field $\mathbb{F}_q$ of q elements. In this note, we classify, up to projective equivalence, hypersurfaces X^n as above which reach two elementary upper bounds for the number of $\mathbb{F}_q$-points on X^n which involve a Thas' invariant.

1. Introduction

Let $\mathbb{F}_q$ be a finite field with q elements, where $q = p^r$ for some prime p and some positive integer r, $\mathbb{P}^{n+1}$ be the projective space over $\mathbb{F}_q$ of dimension $n + 1$, and X^n be a hypersurface in $\mathbb{P}^{n+1}$ defined over $\mathbb{F}_q$ of degree $d \geq 2$ and dimension $n \geq 1$. There is a very rich literature on the general problem of counting or finding bounds on the number of rational points of certain projective varieties defined over finite fields. Since so far only a few things are known about it, many authors addressed their attention to the problem of finding good upper bounds for the number of $\mathbb{F}_q$-points of projective varieties and hypersurfaces defined over a finite field $\mathbb{F}_q$ (see, e.g., [1,2,8,9] and references therein). In particular, several years ago, Thas defined in [13] an invariant k_{X^n} of X^n, that is, the maximum dimension k_{X^n} of an $\mathbb{F}_q$-linear subspace of $\mathbb{P}^{n+1}$ which is contained in X^n, and obtained an upper bound for the number $N_q(X^n)$ of $\mathbb{F}_q$-points of X^n which involved this invariant k_{X^n}. Recently, Homma and Kim established the following elementary upper bound involving k_{X^n} (cf. [7, Theorem 3.2]),

$$N_q(X^n) \leq (d-1)q^{k_{X^n}} N_q(\mathbb{P}^{n-k_{X^n}}) + N_q(\mathbb{P}^{k_{X^n}}) , \tag{1}$$

which is sharp for $k_{X^n} > 0$. Moreover, they proved that (1) is better than Thas' upper bound (see, [7, §7.1]). Finally, in [7] the authors gave the complete list of nonsingular hypersurfaces X^n in $\mathbb{P}^{n+1}$ with n even which reach the equality in (1) for $k_{X^n} = \frac{n}{2}$ (see [7, Theorem 4.1]).

The main purpose of this article is to re-prove in an easy way the Homma–Kim's elementary upper bound (1) for $k_{X^n} > 0$, extending this also to the case $k_{X^n} = 0$, and to give a complete list of hypersurfaces X^n in $\mathbb{P}^{n+1}$ which reach this bound, independently of the parity of n and the singularities of X^n. In particular, observe that $k_{X^n} \leq n$ and note that the right hand of the inequality in (1) increases with k_{X^n}. Thus, the upper bound in (1) reduces to the Segre–Serre–Sørensen's upper bound [10,11] and [12] for the general case $k_{X^n} \leq n$ by replacing n to k_{X^n}, and it becomes the Homma–Kim's elementary bound proved in [5] for the case of hypersurfaces X^n which does not admit $\mathbb{F}_q$-linear components, that is, when $k_{X^n} \leq n-1$, by substituting $n-1$ to k_{X^n}. Furthermore, in both of the above cases, a complete list of hypersurfaces X^n in $\mathbb{P}^{n+1}$ achieving the upper bound in (1) with $k_{X^n} = n, n-1$ is given in [11] and [14], respectively.

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Let W be an algebraic set in $\mathbb{P}^{n+1}$ defined by equations over $\mathbb{F}_q$ and denote by $\mathrm{Sing}(W)(\mathbb{F}_q)$ the set of singular $\mathbb{F}_q$-points of W and by $\mathbb{P}^s * W \subset \mathbb{P}^{n+1}$ the cone over W with vertex $\mathbb{P}^s$ for some $s \in \mathbb{Z}_{\geq 0}$ such that $s \leq n - \dim W$. Therefore, keeping in mind the two above cases, for $0 < k_{X^n} \leq n$ we obtain the following classification result.

Theorem 1 (*Cases* $0 < k_{X^n} \leq n$). *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $d \geq 2$ and dimension $n \geq 1$ defined over $\mathbb{F}_q$. Define*

$$k_{X^n} := \max \left\{ h \mid \text{there exists an } \mathbb{F}_q\text{-linear space } \mathbb{P}^h \subseteq X^n \right\}$$

and suppose that $0 < k_{X^n} \leq n$. Then

$$N_q(X^n) \leq (d-1)q^{k_{X^n}} N_q(\mathbb{P}^{n-k_{X^n}}) + N_q(\mathbb{P}^{k_{X^n}})$$

and equality holds if and only if one of the following possibilities occurs:

(I) $k_{X^n} = n$ *and X^n is a union of d hyperplanes over $\mathbb{F}_q$ that contain a common $\mathbb{F}_q$-linear subspace of codimension 2 in $\mathbb{P}^{n+1}$;*
(II) $0 < k_{X^n} \leq n-1$ *and one of the following cases can occur:*

(1) $d = q+1$ *and X^n is a space-filling hypersurface*

$$(X_0, \ldots, X_{n+1})\, A\, {}^t(X_0^q, \ldots, X_{n+1}^q) = 0,$$

where $A = \left(a_{ij}\right)_{i,j=1,\ldots,n+2}$ is an $(n+2)\times(n+2)$ matrix such that ${}^tA = -A$ and $a_{kk} = 0$ for every $k = 1, \ldots, n+2$; moreover, X^n is nonsingular if and only if $\det A \neq 0$; in particular, if n is odd, then X^n is singular ;

(2) $d = \sqrt{q} + 1$ *and*

(a) $n = 2h$ *with $h \in \mathbb{Z}_{\geq 1}$, $1 \leq k_{X^{2h}} \leq 2h-1$ and one of the following two cases holds:*

(i) *if $\mathrm{Sing}(X^{2h})(\mathbb{F}_q) = \emptyset$, then $k_{X^{2h}} = h$ and X^{2h} is projectively equivalent to a nonsingular Hermitian hypersurface;*
(ii) *if $\mathrm{Sing}(X^{2h})(\mathbb{F}_q) \neq \emptyset$, then $h \geq 2$ and, up to projective equivalence, we have*

$$X^{2h} = \begin{cases} \mathbb{P}^1 * X_H^{2h-2}\,, & k_{X^{2h}} = h+1 \\ \mathbb{P}^3 * X_H^{2h-4}\,, & k_{X^{2h}} = h+2 \\ \ldots & \\ \mathbb{P}^{2h-3} * X_H^{2}\,, & k_{X^{2h}} = h+(h-1)\,; \end{cases}$$

(b) $n = 2h+1$ *with $h \in \mathbb{Z}_{\geq 1}$, $1 \leq k_{X^{2h+1}} \leq 2h$ and, up to projective equivalence, we have*

$$X^{2h+1} = \begin{cases} \mathbb{P}^0 * X_H^{2h}\,, & k_{X^{2h+1}} = h+1 \\ \mathbb{P}^2 * X_H^{2h-2}\,, & k_{X^{2h+1}} = h+2 \\ \mathbb{P}^4 * X_H^{2h-4}\,, & k_{X^{2h+1}} = h+3 \\ \ldots & \\ \mathbb{P}^{2h-2} * X_H^{2}\,, & k_{X^{2h+1}} = h+h\,, \end{cases}$$

where $\mathbb{P}^l * X_H^m \subset \mathbb{P}^{m+l+2}$ is a cone over a nonsingular Hermitian $\mathbb{F}_q$-hypersurface $X_H^m \subset \mathbb{P}^{m+1}$ of dimension m with vertex an $\mathbb{F}_q$-linear subspace $\mathbb{P}^l$;

(3) $d = 2$, $k_{X^n} = \frac{n+h+1}{2} \in \mathbb{Z}_{>0}$ *and X^n is projectively equivalent to a cone $\mathbb{P}^h * Q^{n-h-1} \subset \mathbb{P}^{n+1}$ with vertex an $\mathbb{F}_q$-linear subspace $\mathbb{P}^h$ with $-1 \leq h \leq n-1$, where $Q^{n-h-1} \subset \mathbb{P}^{n-h}$ is the hyperbolic quadric hypersurface*

$$X_0X_1 + X_2X_3 + \cdots + X_{n-h-1}X_{n-h} = 0\,.$$

As to the case $k_{X^n} = 0$, let us recall here that Homma obtained in [4] an upper bound for hypersurfaces $X^n \subset \mathbb{P}^{n+1}$ with $n \geq 1$ without $\mathbb{F}_q$-lines which is valid in general, except for the case $n = 1$ and $d = q = 4$. On the other hand, his bound is better than (1) with $k_{X^n} = 0$. For these reasons, we provide here another elementary upper bound for the number of $\mathbb{F}_q$-points of hypersurfaces X^n in $\mathbb{P}^{n+1}$ with $k_{X^n} = 0$ for any $n \geq 1$ and we characterize those X^n which achieve this bound in the following result.

Theorem 2 (*Case* $k_{X^n} = 0$). *Let $X^n \subset \mathbb{P}^{n+1}$ be a hypersurface of degree $d \geq 2$ and dimension $n \geq 1$ defined over $\mathbb{F}_q$. If $k_{X^n} = 0$, then*

$$N_q(X^n) \leq (d-1)q^n + (d-2)N_q(\mathbb{P}^{n-1}) + 1$$

and equality holds if and only if $d = 2$ and, up to projective equivalence, either $n = 1$ and $X^1 :\ X_0^2 + X_1^2 + X_2^2 = 0$ is a nonsingular plane conic, or $n = 2$ and $X^2 :\ f(X_0, X_1) + X_2X_3 = 0$ is a nonsingular elliptic surface, where $f(X_0, X_1) = \alpha X_0^2 + X_0X_1 + X_1^2$ is an irreducible binary quadratic form with $\alpha \in \{t \in \mathbb{F}_q \mid t + t^2 + t^4 + \cdots + t^{2^{r-1}} = 1\}$ if $q = 2^r$ for some $r \in \mathbb{Z}_{\geq 1}$ and such that $1 - 4\alpha$ is a non-square if q is odd.

Finally, in Corollary 9 of Section 3 we give an immediate consequence of Theorems 1 and 2 for the nonsingular case.