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Graphs with maximum average degree less than $\frac{11}{4}$ are (1, 3)-choosable

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ABSTRACT

The well known 1–2–3-Conjecture asserts that every connected graph G with at least three vertices can be edge weighted with 1, 2, 3, so that for any two adjacent vertices u and v, the sum of the weights of the edges incident to u is distinct from the sum of the weights of the edges incident to v. In this paper, we consider the list version of this problem and prove that graphs with maximum average degree smaller than $\frac{11}{4}$ are strongly $(1, 3)$-choosable, which implies that the 1–2–3 conjecture is true for such graphs. This improves the results in Cranston et al. (2014)[7] and Przybyło et al. (2017).

1. Introduction

Suppose $G = (V, E)$ is a graph. For each $z \in V(G) \cup E(G)$, let x_z be a variable associated to z. Let $E_G(v)$ be the set of edges incident to v. Fix an arbitrary orientation D of G. Define polynomial $P_G(\{x_z : z \in V(G) \cup E(G)\})$ as follows:

$$P_G(\{x_z : z \in V(G) \cup E(G)\}) = \prod_{uv \in E(D)} \left(\left(\sum_{e \in E(v)} x_e + x_v \right) - \left(\sum_{e \in E(u)} x_e + x_u \right) \right).$$

An *index function* of G is a mapping η which assigns to each vertex or edge z of G a non-negative integer $\eta(z)$. An index function η of G is *valid* if $\sum_{z \in V \cup E} \eta(z) = |E|$. Note that polynomial P_G has degree $|E|$. For a valid index function η, let $c_{\eta,G}$ be the coefficient of the monomial $\prod_{z \in V \cup E} x_z^{\eta(z)}$ in the expansion of P_G.

Definition 1. Assume G is a graph and η is an index function of G. If there is a valid index function η' such that $\eta'(z) < \eta(z)$ for all z, and $c_{\eta',G} \neq 0$, then we say G is *strongly η-choosable*. The index function η' is called a *witness of G being strongly η-choosable*.

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The concept of strongly η-choosability is motivated by total weighting of graphs.

Assign a real number $\phi(z)$ to the variable x_z, and view $\phi(z)$ as the weight of z. Let $P_G(\phi)$ be the evaluation of the polynomial at $x_z = \phi(z)$. We say ϕ is a *proper total weighting* of G if $P_G(\phi) \neq 0$. In other words, ϕ is a proper total weighting of G if for any two adjacent vertices u and v, $\sum_{e\in E(u)}\phi(e) + \phi(u) \neq \sum_{e\in E(v)}\phi(e) + \phi(v)$.

If ϕ is a proper total weighting of G such that $\phi(e) = 0$ for all edges e, then ϕ is simply a proper vertex colouring of G. If $\phi(v) = 0$ for all vertices v, and $\phi(e) \in \{1, 2, \ldots, k\}$ for all edges e, then ϕ is called a vertex colouring k-edge weighting of G. The well-known 1–2–3-conjecture, proposed in [10], says that every graph with no isolated edge has a vertex colouring 3-edge weighting. This conjecture remains open. A sequence of papers made progress on this problem and the best result on this conjecture, proved in [9], is that every graph with no isolated edges has a vertex-colouring 5-edge weighting. The 1–2 conjecture, proposed by Przybyło and Woźniak in [12], asserts that every graph G has a proper total weighting ϕ with $\phi(z) \in \{1, 2\}$ for all $z \in V(G) \cup E(G)$. The best result on this conjecture is that every graph G has a proper total weighting ϕ with $\phi(v) \in \{1, 2\}$ for $v \in V(G)$ and $\phi(e) \in \{1, 2, 3\}$ for $e \in E(G)$ [8].

For an index function η, we say G is *η-choosable*, if for any total list assignment L with $|L(z)| = \eta(z)$, G has a proper total weighting ϕ and $\phi(z) \in L(z)$ for all $z \in V \cup E$.

It follows from the Combinatorial Nullstellensatz [3,5] that if G is strongly η-choosable, then G is η-choosable. Indeed, strongly η-choosable means "η-choosable that can be proved by using Combinatorial Nullstellensatz".

We say a graph is (strongly) (k, k')-choosable if it is (strongly) η-choosable, where $\eta(v) = k$ for each vertex v and $\eta(e) = k'$ for each edge e. Note that $(k, 1)$-choosable is equivalent to vertex k-choosable.

As a strengthening of the 1–2–3-conjecture, it was conjectured in [19] that every graph with no isolated edges is $(1, 3)$-choosable (or even strongly $(1, 3)$-choosable). As a strengthening of the analog 1–2-conjecture proposed by Przybyło and Woźniak [13], it was proposed in [19] that every graph is $(2, 2)$-choosable (or even strongly $(2, 2)$-choosable).

There are many partial results on the 1–2–3 conjecture and on the total weight choosability conjectures [1,2,6,7,9–20]. It was shown in [20] that every graph is $(2, 3)$-choosable. However, it is unknown whether there is a constant k such that every graph with no isolated edge is $(1, k)$-choosable. It is also unknown whether there is a constant k such that every graph is $(k, 2)$-choosable.

Some special graphs are shown to be $(1, 3)$-choosable, such as complete graphs, complete bipartite graphs, trees [6], outerplanar graphs [18], Cartesian product of an even number of even cycles, of a path and an even cycle, of two paths [16]. Some special graphs are shown to be $(2, 2)$-choosable, such as complete graphs [19], outerplanar graphs [18], complete bipartite graphs [17], subcubic graphs, Halin graphs [21], 2-degenerate graphs [18].

It was proved in [10] and [12] that 1–2–3 conjecture and 1–2 conjecture hold for 3-colourable graph. For sparse graphs with $\mathrm{mad}(G) < \frac{5}{2}$, it was proved in [11] that $(2, 2)$-choosable conjecture holds, and $(1, 3)$-choosable conjecture holds for some special list assignments (namely, for those L with $L(v) = \{0\}$ for each vertex v and $L(e)$ contains three *positive* numbers for each edge e).

In this paper, we prove the following result.

Theorem 2. *Let G be a graph without an isolated edge and $\mathrm{mad}(G) < \frac{11}{4}$. Then G is strongly $(1, 3)$-choosable.*

2. Unavoidable configurations

The proof of Theorem 2 uses the discharging method. We shall give a list of configurations. In this section, we prove that a graph G with $\mathrm{mad}(G) < \frac{11}{4}$ contains at least one of the listed configurations. In the next section, we prove that any minimal counterexample to Theorem 2 does not contain any of the listed configurations, and hence completes the proof of Theorem 2.

We first give some notations. A vertex of degree k (respectively, at least k or at most k) is called as k-vertex (respectively, a k^+-vertex, or a k^--vertex). A *k-neighbour* of a vertex v is a neighbour of v which is a k-vertex.

Let $d_2(v)$ be the number of 2-neighbours of v. A *weak* (respectively, *very weak*) vertex is a 3-vertex v with $d_2(v) \geq 1$ (respectively, $d_2(v) = 2$). A *weak neighbour* (respectively, a *very weak neighbour*) of a vertex v is a neighbour of v which is a weak vertex (respectively, very weak vertex).

A vertex v is *poor* if one of the following holds:

- $d_2(v) \geq d(v) - 1$.
- v is a weak vertex and has a weak neighbour.

A vertex is *rich* if it is not poor.

Lemma 3. *If G is a bipartite graph with $\mathrm{mad}(G) < \frac{11}{4}$, and with minimum degree at least 2, then it contains at least one of the following configurations:*

- *$(C1)$ A vertex whose neighbours are all 2-vertices.*
- *$(C2)$ A k-vertex with $(k-1)$ 2-neighbours and one weak neighbour.*
- *$(C3)$ A 2-vertex adjacent to a 2-vertex or adjacent to two poor vertices.*
- *$(C4)$ A 3-vertex with three weak neighbours or with one weak neighbour and one very weak neighbour.*
- *$(C5)$ A 4-vertex with two 2-neighbours and two weak neighbours.*