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# Weighted Hamming metric structures

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ABSTRACT

It is known that the binary generalized Goppa codes are perfect codes for the weighted Hamming metrics. In this paper, we present the existence of a weighted Hamming metric that admits a binary Hamming code (*resp.* an extended binary Hamming code) to be perfect code. For a special weighted Hamming metric, we also give some structures of a 2-perfect code, show how to construct a 2-perfect linear code and obtain the weight distribution of a 2-perfect code from the partial information of the code.



## 1. Introduction

There have been attempts to consider coding theory not only for the Hamming metric but also for other metrics [1,3,5,6]. Also, many efforts have been made to construct a good code that can correct errors. Therefore, many types of perfect codes exist [1,3,5,6]. However, many results in coding theory are codes for channels in which the error is consistent with the Hamming metric, but in some channels the distribution of errors among the codeword positions is nonuniform. In addition, nontrivial perfect binary codes are rare for the Hamming metric. The Hamming codes and the Golay codes are the only nontrivial perfect linear codes for the Hamming metric. So S. Bezzateev and N. Shekhunova [1] considered a perfect code for a weighted Hamming metric to apply codes to channels with the distribution of errors which is nonuniform. The authors gave some basic properties of a perfect code for a weighted Hamming metric. Of particular interest is that a binary generalized Goppa code is a perfect code for a certain weighted Hamming metric. Heden [2] studied a generalization of 1-perfect code for the Hamming metric. It is thus natural to study the weight structure that admits a binary Hamming code (*resp.* an extended binary Hamming code) to be a perfect code and find other perfect codes.

In this paper, we study perfect codes in $\mathbb{F}_2^n$ equipped with a special metric called weighted Hamming metric. Some perfect codes are known but the information about the codes for a weighted Hamming metric is rare. In Section 2, we show that the binary Hamming code $\mathcal{H}_m$ (*resp.* the extended binary Hamming code $\widetilde{\mathcal{H}}_m$) is 2-perfect (*resp.* 2- or 3-perfect) for a particular weighted Hamming metric. In Section 3, with a particular weighted Hamming metric, we deduce how to construct a linear 2-perfect code and give the properties of a 2-perfect code. In the next, we generalize the concept of a 2-perfect code and present a weight distribution of the code for the metric.

Let $\mathbb{F}_2$ be a finite field of order two and $\mathbb{F}_2^n$ a vector space of binary $n$-tuples.

**Definition 1.1** (*[1]*). Let $\pi_i \in \mathbb{N}$ be a weight of position $i$ and $\pi = (\pi_1, \dots, \pi_n)$ a weight vector of length $n$, respectively.
The $\pi$-weight $w_\pi$ of a vector $\mathbf{x} = (x_1, \dots, x_n)$ of $\mathbb{F}_2^n$ is defined by the function

$$w_\pi(x) = \sum_{i=1}^{n} \pi_i \cdot x_i,$$

and the $\pi$-distance $d_\pi(\mathbf{x}, \mathbf{y})$ between vectors $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{F}_2^n$ is defined as $d_\pi(\mathbf{x}, \mathbf{y}) = w_\pi(\mathbf{x} - \mathbf{y})$.

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**Lemma 1.2** *([6, Lemma 1.1]). Let $\pi$ be a weight vector of length n. Then, the $\pi$-distance $d_\pi$ is a metric on $\mathbb{F}_2^n$.*

For a weight vector $\pi$ of length $n$, the $\pi$-metric $d_\pi$ is also called *a weighted Hamming metric* with weight vector $\pi$ on $\mathbb{F}_2^n$. In particular, if $\pi$ is the all-one vector $(1, 1, \ldots, 1)$, then the $\pi$-weight $w_\pi$ and $\pi$-metric $d_\pi$ become the Hamming weight $w_H$ and the Hamming distance $d_H$ in the classical coding theory, respectively.

Let $\mathbf{x}$ be a vector of $\mathbb{F}_2^n$ and $r$ a nonnegative integer. For a given weight vector $\pi$ of length $n$, the *$\pi$-sphere* with center $\mathbf{x}$ and radius $r$ is defined as the set

$$S_\pi(\mathbf{x}; r) = \{\mathbf{y} \in \mathbb{F}_2^n \mid d_\pi(\mathbf{x}, \mathbf{y}) \leq r\}.$$

Note that $S_\pi(x; r) \subseteq S_H(x; r)$. Since every 1-perfect code for the Hamming metric is known, we only consider an $r$-perfect code ($r \geq 2$) for a weighted Hamming metric.

For a subset $C \subset \mathbb{F}_2^n$ and a given weight vector $\pi$ of length $n$, we call $C$ a *$\pi$-code of length n*. For a $\pi$-code $C$, let the minimum $\pi$-distance $d_\pi(C)$ of $C$ be the minimum $\pi$-distance between distinct codewords of $C$. We now introduce the definition of perfect code for a weighted Hamming metric and present a sufficient and necessary condition for a given code to be a perfect code.

The *support* of a vector $\mathbf{x}$ of $\mathbb{F}_2^n$ is the set of nonzero coordinate positions of $\mathbf{x}$, so the size of the support of $\mathbf{x}$ is also $w_H(\mathbf{x})$. Throughout this paper, we identify a vector $\mathbf{x}$ of $\mathbb{F}_2^n$ with its support. For vectors $\mathbf{x}, \mathbf{y}_1, \ldots, \mathbf{y}_m$ of $\mathbb{F}_2^n$, let $\{\mathbf{y_1}, \ldots, \mathbf{y_m}\}$ be a partition of $\mathbf{x}$ if the disjoint union of the elements of $\{\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_m}\}$ is equal to the support of $\mathbf{x}$. For example, we identify $\mathbf{x} = (1, 0, 0, 1)$ with a set $\{1, 4\}$ and have $\{(1, 0, 0, 0), (0, 0, 0, 1)\}$ as a partition of $\mathbf{x}$.

**Definition 1.3** ([6]). Let $\pi$ be a weight vector of length $n$ and $C$ a $\pi$-code of length $n$. We say that a code $C$ is an $r$-perfect $\pi$-code if the disjoint union of the spheres $S_\pi(\mathbf{c}; r)$ centered at $\mathbf{c} \in C$ and with radius $r$ is equal to $\mathbb{F}_2^n$.

**Proposition 1.4** *([6]). Let $\pi$ be a weight vector of length n and C an $[n, k]$ binary linear $\pi$-code. Then, C is an r-perfect $\pi$-code if and only if the following two conditions are satisfied:*

(1) *(The sphere packing condition) $|S_\pi(\mathbf{0}; r)| = 2^{n-k}$,*
(2) *(The partition condition) for any non-zero codeword $\mathbf{c}$ and any partition $\{\mathbf{x}, \mathbf{y}\}$ of $\mathbf{c}$, either $w_\pi(\mathbf{x}) \geq r + 1$ or $w_\pi(\mathbf{y}) \geq r + 1$.*

## 2. Perfect $\pi$-codes $\mathcal{H}_m$ and $\widetilde{\mathcal{H}}_m$

For the Hamming metric, a binary Hamming code is a 1-perfect code, but an extended binary Hamming code is not a perfect code. In this section, we introduce the weighted Hamming metric for which the binary Hamming code $\mathcal{H}_m$ (*resp.* the extended binary Hamming code $\widetilde{\mathcal{H}}_m$) becomes a perfect code. We also give an example.

**Definition 2.1.** Let $\widetilde{H}_m (m \geq 2)$ be an $(m + 1) \times 2^m$ binary matrix whose first row is the all-one vector $(1, \ldots, 1)$ of length $2^m$ and the remaining $m$ rows of $\widetilde{H}_m$ form a $m \times 2^m$ submatrix whose $i$th column corresponds to the 2-adic representation of $i - 1$. By deleting the first row and the first column of $\widetilde{H}_m$, we have an $m \times (2^m - 1)$ binary matrix $H_m$. Let $\mathcal{H}_m$ be the binary Hamming code of length $2^m - 1$ with the minimum distance 3 with the parity-check matrix $H_m$ and $\widetilde{\mathcal{H}}_m$ the extended binary Hamming code of length $2^m$ with the minimum distance 4 with the parity-check matrix $\widetilde{H}_m$ for the Hamming metric.

Denote the set of coordinates of $\pi$-weight $i$ by $S_i$ and the size of $S_i$ by $s_i$ for $i \in \mathbb{N}$. We identify a weight vector $\pi$ with the set of $S_i$ for $i \in \mathbb{N}$. For example, we identify $\pi = (1, 2, 4, 1)$ with $\{S_1 = \{1, 4\}, S_2 = \{2\}, S_4 = \{3\}\}$.

### *2.1. 2-perfect $\pi$-codes $\mathcal{H}_m$ and $\widetilde{\mathcal{H}}_m$*

In the first part of this subsection, we give a sufficient condition for the existence of a weighted Hamming metric which admits the binary Hamming code $\mathcal{H}_m$ to be a 2-perfect code. In the second part, we classify all weight vectors which admit the extended binary Hamming code $\widetilde{\mathcal{H}}_m$ to be 2-perfect.

By Proposition 1.4, if the binary Hamming code $\mathcal{H}_m (m \geq 2)$ is a 2-perfect $\pi$-code for a given weight vector $\pi$, then

$$2^m = |S_\pi(\mathbf{0}; 2)| = 1 + s_1 + \binom{s_1}{2} + s_2.$$

Thus we have $s_2 = 2^m - (1 + s_1 + \binom{s_1}{2})$.

For a set $X$ and a positive integer $r$, we denote a set of all $r$-subset of $X$ by $\binom{X}{r}$. We also denote a set of all natural numbers which are less than or equal to $n$ by $[n]$.

**Theorem 2.2.** *For an integer $m \geq 2$, there exists a weight vector $\pi$ of length $2^m - 1$ which admits the binary Hamming code $\mathcal{H}_m (m \geq 2)$ to be a 2-perfect $\pi$-code provided that*

$$1 + \binom{s_1 - 1}{1} + \binom{s_1 - 1}{2} + \binom{s_1 - 1}{3} < 2^m, \quad (1)$$

*where the integer $s_1$ is the size of $S_1$ which is the set of coordinates of $\pi$-weight 1.*