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A new family of Geodesic transitive graphs☆

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ABSTRACT

We present a new family of locally geodesic transitive graphs with arbitrarily large diameter and valencies, containing a particular case to be geodesic transitive. We also prove that it is a unique family in some generalised family of graphs.

1. Introduction

In this paper, we mainly construct a new family of locally geodesic transitive graphs with arbitrarily large diameter and valencies, which indeed, as we know, is the *first* family of locally geodesic transitive graphs being *not* vertex transitive. We also present some more properties of these graphs.

Let Γ be a finite, simple and undirected graph with vertex set Ω and edge set E. A sequence of $s+1$ distinct vertices of Γ, say $(v_0 = x, \dots, v_s = y)$ such that consecutive vertices are adjacent is called a *path* of length s from x to y. When the limitation of all the vertices being distinct in this sequence is simply replaced by $v_{i-1} \neq v_{i+1}$ for $0 < i < s$, it is called an *s-arc* instead. If a path of length r from x to y is a shortest path between the two vertices, we usually write $r = \mathrm{d}_\Gamma(x, y)$ or simply $r = \mathrm{d}(x, y)$, and call it the *distance* between x and y; meanwhile call the path an *r-geodesic*. Clearly geodesics can be seen as special arcs. The largest distance between vertex pairs is called the *diameter* of Γ, denoted by $\mathsf{diam}(\Gamma)$. For $1 \leqslant i \leqslant \mathsf{diam}(\Gamma)$ and an arbitrary vertex α, define $\Gamma_i(\alpha) := \{v \in \Omega \mid \mathrm{d}(\alpha, v) = i\}$. In particular, $\Gamma(\alpha) := \Gamma_1(\alpha)$ is called the *neighbourhood* of α, the size of which is called the *valency* of α. In addition, the length of the shortest cycle in a graph is called its *girth*.

Denote by $\mathrm{Aut}\Gamma$ the *automorphism group* of Γ, which contains all the permutations of the vertices that preserve the adjacency. Notice $\mathrm{Aut}\Gamma$ also preserving a geodesic or an arc. Now it is ready to introduce some relevant graph symmetry properties: Γ is called *locally* (G, s)*-distance transitive*, called *locally* (G, s)*-geodesic transitive*, or called *locally* (G, s)*-arc transitive*, if there exists a subgroup G of $\mathrm{Aut}\Gamma$ that for $i \leqslant s \leqslant \mathsf{diam}(\Gamma)$, the stabiliser of any vertex α in G denoted by G_α acts transitively on $\Gamma_i(\alpha)$, transitively on the set of all i-geodesics starting at α, or transitively on the set of all i-arcs starting at α, respectively. (One may find different definitions of these properties otherwhere, which will be explained in our **Remarks**.) Moreover, if G itself acts transitively on the vertex set, we remove the word *locally* from them. As well as when we do not emphasise the group G, it is also allowed to be removed. Particularly for symmetry properties related to distance or geodesic, if such a property is satisfied for all s up to $\mathsf{diam}(\Gamma)$, we also remove the prefix s. Here we take *G-geodesic transitive* as an example, which means locally $(G, \mathsf{diam}(\Gamma))$-geodesic transitive and also G-vertex transitive.

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Table 1
$\#(\Gamma_i(v))$ for $\Gamma_q^n(k-1,k)$, $n \geqslant 2k$.

	$\#(\Gamma_{2i}(v)),\ 1 \leqslant i \leqslant k-1$	$\#(\Gamma_{2i+1}(v)),\ 1 \leqslant i \leqslant k-1$
$v \in \begin{bmatrix} V \\ k-1 \end{bmatrix}$	$q^{i^2} \cdot \begin{bmatrix} k-1 \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k+1 \\ i \end{bmatrix}_q$	$q^{i(i+1)} \cdot \begin{bmatrix} k-1 \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k+1 \\ i+1 \end{bmatrix}_q$
$v \in \begin{bmatrix} V \\ k \end{bmatrix}$	$q^{i^2} \cdot \begin{bmatrix} k \\ i \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q$	$q^{i(i+1)} \cdot \begin{bmatrix} k \\ i+1 \end{bmatrix}_q \cdot \begin{bmatrix} n-k \\ i \end{bmatrix}_q$

There have been longtime interest and a great deal of mathematical effort in studying these graph symmetry properties, for instance seeing [4, Chap.4] and [3,9–11]. Among these properties, the transitivity on geodesics is a recent topic, while the other two have been studied more explicitly. As an example, the finite 2-geodesic transitive graphs of different valencies have been characterised lately, in a series of papers including [1,5,6]. In fact, it is not an easy task to construct a specific family of geodesic transitive graphs, especially with unbounded diameter and valency. Most of the cases we have seen are some well-known graphs, namely, Johnson Graphs, Hamming Graphs, Odd Graphs and so on, seeing [8]. Meanwhile we have not seen any example of locally geodesic transitive graphs to be *not* vertex transitive. In this paper we will present a new family of locally geodesic transitive graphs, which seem not so familiar and contain only a particular case to be vertex transitive (hence geodesic transitive). As we know, it gives such an example for the first time.

To do that, firstly, let V be a vector space over field $\mathbb{F}_q$ of dimension n, and $\begin{bmatrix} V \\ i \end{bmatrix}$ denotes the set of all subspaces of V of dimension i. For $0 < \ell < k < n$, define Incident Graphs $\Gamma_q^n(\ell, k)$ to be a bipartite graph with vertex set $\begin{bmatrix} V \\ \ell \end{bmatrix} \bigcup \begin{bmatrix} V \\ k \end{bmatrix}$ that any two vertices $X \in \begin{bmatrix} V \\ \ell \end{bmatrix}$ and $Y \in \begin{bmatrix} V \\ k \end{bmatrix}$ are adjacent whenever $X \subseteq Y$. Then we can describe our main results in the following:

Theorem 1.1. *The graph $\Gamma_q^n(k-1, k)$ is of girth six and locally distance transitive; moreover, it is locally geodesic transitive. In particular for $n = 2k-1$, the graph $\Gamma_q^{2k-1}(k-1, k)$ is geodesic transitive.*

Corollary 1.2. *The graph $\Gamma_q^n(k-1, k)$ is locally 3-arc transitive and for $n \geqslant 4$ it is locally 4-geodesic transitive but not locally 4-arc transitive; in particular, the graph $\Gamma_q^{2k-1}(k-1, k)$ is 3-arc transitive and for $k \geqslant 3$ it is 4-geodesic transitive but not 4-arc transitive.*

Remarks.

1. The girth is the key point to link geodesic transitivity to arc transitivity, thus we emphasis it in the main theorem. In fact, by our definition being (locally) s-arc transitive implies being (locally) s-geodesic transitive; meanwhile in the opposite direction, it is proved that there exist infinitely many 3-geodesic transitive but not 3-arc transitive graphs with arbitrarily large diameter and valency [7, Theorem 1.1]. As Corollary 1.2 tells, infinite family of graphs $\Gamma_q^{2k-1}(k-1, k)$ $(k \geqslant 3)$ also with arbitrarily large diameter and valency, give examples of being 4-geodesic transitive but not 4-arc transitive.

2. For geodesic transitive case $n = 2k-1$, we are curious about the possible relationships between the graph $\Gamma_q^{2k-1}(k-1, k)$ and other well-known graphs. It is only partly solved in Section 2.

3. Some researchers define locally transitive properties in a different way as we mentioned. For example, being locally (G, s)-arc transitive requires the stabiliser G_α of any fixed vertex α being transitive only on the set of all s-arcs starting at α (rather than i-arcs for $i \leqslant s$). Note that locally transitive properties are not monotone properties: if G_α is transitive on the set of all s-arcs or all s-geodesics starting at α, it does not follow that G_α is also transitive on shorter arcs or geodesics starting at α (for more seeing [2]). We declare the two kinds of definitions being surely not equivalent.

This paper is organised as follows. In Section 2 we prove the main results Theorem 1.1, Corollary 1.2 and present some more properties of the new graphs. In Section 3, we extend the definition of Incident graphs, so we can regard the graph $\Gamma_q^n(k-1, k)$ as a unique case among some generalised family of graphs, seeing Theorem 3.1.

2. Proof of the main results

In this section we prove Theorem 1.1 and Corollary 1.2 step by step. Armed with some necessary lemmas, we prove in order that the graph $\Gamma_q^n(k-1, k)$ defined as above is locally distance transitive (Theorem 2.3); is locally geodesic transitive (Theorem 2.5); is vertex transitive for $n = 2k-1$ (Theorem 2.6); and is of girth six (Theorem 2.7). In addition, we calculate $\#(\Gamma_i(v))$ for an arbitrary vertex v and $i \leqslant \mathrm{diam}(\Gamma)$ in Table 1. At last we present how the graph $\Gamma_q^n(k-1, k)$ induces another well-known family of graphs called Grassmann Graphs; as a corollary, we know that Grassmann graphs are also geodesic transitive (Corollary 2.9).

Firstly, the general linear group $\mathrm{GL}(V)$ acts on $\begin{bmatrix} V \\ i \end{bmatrix}$ naturally: for any $U \in \begin{bmatrix} V \\ i \end{bmatrix}$ and any $g \in \mathrm{GL}(V)$, $U^g := \{u^g \mid u \in U\}$ is still a subspace of dimension i. Furthermore, let $\pi : \mathrm{GL}(V) \to \mathrm{PGL}(V)$ be the natural homomorphism, where $\mathrm{PGL}(V)$ is the projective general linear group. Then for $0 < \ell < k < n$, $\mathrm{PGL}(V)$ acts on $\begin{bmatrix} V \\ \ell \end{bmatrix} \bigcup \begin{bmatrix} V \\ k \end{bmatrix}$ faithfully, with $U^{\pi(g)} = U^g$. Meanwhile