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New strongly regular graphs from orthogonal groups $O^+(6, 2)$ and $O^-(6, 2)$

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A B S T R A C T

We prove the existence of strongly regular graphs with parameters (216, 40, 4, 8) and (540, 187, 58, 68). We also construct a strongly regular graph with parameters (540, 224, 88, 96) that was previously unknown. Further, we construct all distance-regular graphs with at most 600 vertices, admitting a transitive action of the orthogonal group $O^+(6, 2)$ or $O^-(6, 2)$. Furthermore, we show that under certain conditions an orbit matrix M of a strongly regular graph Γ can be used to define a new strongly regular graph $\widetilde{\Gamma}$, where the vertices of the graph $\widetilde{\Gamma}$ correspond to the orbits of Γ (the rows of M). We show that some of the obtained strongly regular graphs are related to each other in a way that one can be constructed from an orbit matrix of the other.

1. Introduction

We assume that the reader is familiar with the basic facts of group theory, theory of strongly regular graphs (SRGs) and theory of distance-regular graphs (DRGs). We refer the reader to [6,17] for relevant background reading in group theory, to [3,21] for theory of strongly regular graphs, and to [5,10] for theory of distance-regular graphs.

The orthogonal groups $O^+(6, 2)$ and $O^-(6, 2)$ are simple groups of orders 20160 and 25920, respectively. It is well known (see [6]) that $O^+(6, 2) \cong A_8 \cong L(4, 2)$ and $O^-(6, 2) \cong O(5, 3) \cong U(4, 2) \cong S(4, 3)$. Using the method outlined in [9] we classify SRGs with at most 600 vertices admitting a transitive action of the group $O^+(6, 2)$ or $O^-(6, 2)$. Among others, we construct first examples of SRGs with parameters (216, 40, 4, 8) and (540, 187, 58, 68), *i.e.* we proved the existence of SRGs with these parameters. Further, one of the constructed SRGs with parameters (540, 224, 88, 96) is new to the best of our knowledge. We also classify DRGs having diameter $d \geq 3$ with at most 600 vertices admitting a transitive action of $O^+(6, 2)$ or $O^-(6, 2)$. The classification, that was conducted by use of computers, was computationally demanding. The running time complexity of the algorithm used for the construction of graphs depends on the number of parameters, such as the size of the used subgroup, the number of orbits of a block stabilizer (in $O^+(6, 2)$ or $O^-(6, 2)$), the number of vertices of the graphs. The most demanding problem, that we solved completely, was the attempt to construct a SRG on 560 vertices with $k = 273$. To complete the exhaustive search in that case, we had to analyse 351 481 264 possibilities for the first row of the adjacency matrix.

Further, we introduce a method of constructing new SRGs from orbits, *i.e.* orbit matrices, of known SRGs. We apply this method to the SRGs obtained from $O^+(6, 2)$ and $O^-(6, 2)$. To find the graphs and compute their full automorphism groups, we used programmes written for Magma [2] and GAP [11]. The constructed SRGs and DRGs can be found at the link: http://www.math.uniri.hr/~asvob/SRGs_DRGs.txt.

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Table 1
Subgroups of the group $U(4,2) \cong O^-(6,2)$.

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^2	$2^4 : A_5$	960	27	3	yes
H_2^2	S_6	720	36	3	yes
H_3^2	$3_+^{1+2} : 2A_4$	648	40	3	yes
H_4^2	$3^3 : S_4$	648	40	3	yes
H_5^2	$2^{\cdot}(A_4 \times A_4).2$	576	45	3	yes
H_6^2	$(3^2 : 3) : 8$	216	120	7	no
H_7^2	$2_+^{1+4} : S_3$	192	135	9	no
H_8^2	S_5	120	216	10	no
H_9^2	$2A_4 : 2$	48	540	25	no

Table 2
SRGs constructed from the group $U(4,2) \cong O^-(6,2)$.

Graph Γ	Parameters	$Aut(\Gamma)$	Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^2 = \Gamma(U(4,2), H_1^2)$	(27, 10, 1,5)	$U(4,2):2$	$\Gamma_7^2 = \Gamma(U(4,2), H_7^2)$	(135, 64, 28, 32)	$O^+(8,2):2$
$\Gamma_2^2 = \Gamma(U(4,2), H_2^2)$	(36, 15, 6, 6)	$U(4,2):2$	$\Gamma_8^2 = \Gamma(U(4,2), H_8^2)$	(216, 40, 4, 8)	$U(4,2):2$
$\Gamma_3^2 = \Gamma(U(4,2), H_3^2)$	(40, 12, 2, 4)	$U(4,2):2$	$\Gamma_9^2 = \Gamma(U(4,2), H_9^2)$	(540, 187, 58, 68)	$2 \times (U(4,2):2)$
$\Gamma_4^2 = \Gamma(U(4,2), H_4^2)$	(40, 12, 2, 4)	$U(4,2):2$	$\Gamma_{10}^2 = \Gamma(U(4,2), H_9^2)$	(540, 187, 58, 68)	$2 \times U(4,2)$
$\Gamma_5^2 = \Gamma(U(4,2), H_5^2)$	(45, 12, 3, 3)	$U(4,2):2$	$\Gamma_{11}^2 = \Gamma(U(4,2), H_9^2)$	(540, 224, 88, 96)	$2 \times U(4,2)$
$\Gamma_6^2 = \Gamma(U(4,2), H_6^2)$	(120, 56, 28, 24)	$O^+(8,2):2$	$\Gamma_{12}^2 = \Gamma(U(4,2), H_9^2)$	(540, 224, 88, 96)	$U(4,3):D_8$

2. SRGs from the groups $O^+(6,2)$ and $O^-(6,2)$

In this section we classify SRGs with at most 600 vertices admitting a transitive action of the orthogonal group $O^+(6,2)$ or $O^-(6,2)$. The graphs are constructed using the method described in [9]. Using this method one can construct all regular graphs admitting a transitive action of the group G, but we will be interested only in those regular graphs that are distance-regular, and especially strongly regular. The construction of graphs admitting a transitive action of $O^-(6,2)$ leads to the discovery of the first examples of SRGs with parameters (216, 40, 4, 8) and (540, 187, 58, 68) and a new SRG with parameters (540, 224, 88, 96).

2.1. SRGs from $O^-(6,2)$

The unitary group $U(4,2)$ is the simple group of order 25920, and up to conjugation it has 116 subgroups. There are 40 subgroups of the group $U(4,2)$ up to index 600. In Table 1 we give the list of all the subgroups $H_i^2 \leq U(4,2)$ which lead to the construction of SRGs.

Using the method described in [9] we obtained all SRGs on which the unitary group $U(4,2)$ acts transitively and with at most 600 vertices. Using the computer search we obtained SRGs on 27, 36, 40, 45, 120, 135, 216 or 540 vertices. Finally, we determined the full automorphism groups of the constructed SRGs.

Theorem 1. *Up to isomorphism there are exactly* 12 *strongly regular graphs with at most* 600 *vertices, admitting a transitive action of the group* $U(4,2)$*. These strongly regular graphs have parameters* (27, 10, 1, 5)*,* (36, 15, 6, 6)*,* (40, 12, 2, 4)*,* (45, 12, 3, 3)*,* (120, 56, 28, 24)*,* (135, 64, 28, 32)*,* (216, 40, 4, 8)*,* (540, 187, 58, 68) *and* (540, 224, 88, 96)*. Details about the obtained strongly regular graphs are given in Table 2.*

From [3,4] we deduce that our SRG with parameters (216, 40, 4, 8) and two SRGs with parameters (540, 187, 58, 68) are the first known examples of SRGs with these parameters. Moreover, the constructed SRG(216, 40, 4, 8) is the first known SRG on 216 vertices. Further, the graph Γ_{11}^2 was to the best of our knowledge previously unknown.

Remark 1. The strongly regular graphs Γ_4^1 and Γ_6^2 with parameters (120, 56, 28, 24) obtained from the groups A_8 and $U(4,2)$, respectively, are isomorphic.

The SRGs with parameters (27, 10, 1, 5), (36, 15, 6, 6), (40, 12, 2, 4) and (45, 12, 3, 3) are completely classified (see [3,7,16,19]). The graphs Γ_1^2, Γ_2^2, Γ_3^2, Γ_4^2 and Γ_5^2 are rank 3 graphs, where Γ_1^2 is the unique SRG on 27 vertices. Note that Γ_3^2 and Γ_4^2 are point graphs of generalized quadrangles GQ(3, 3) (see [12]), and Γ_4^2 corresponds to the point-hyperplane design in the projective geometry PG(3, 3). The graph Γ_5^2 is the only vertex-transitive SRG with parameters (45, 12, 3, 3). Further, Γ_7^2 is the complementary graph of the polar graph $O^+(8,2)$, and Γ_{12}^2 is the polar graph $NU(4,3)$.

Remark 2. SRGs can be constructed as point graphs of partial geometries (see [20]). In particular, the existence of a partial geometry pg(11, 16, 4) would imply the existence of a SRG(540, 187, 58, 68), but there is no known example of a partial geometry with these parameters. If a strongly regular graph Γ with parameters (540, 187, 58, 68) is obtained from a partial geometry pg(11, 16, 4), then a line in a pg(11, 16, 4) corresponds to a clique of size 12 in Γ. Since a pg(11, 16, 4)