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Domination and upper domination of direct product graphs

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ABSTRACT

Let $X_{\mathbb{Z}/n\mathbb{Z}}$ denote the unitary Cayley graph of $\mathbb{Z}/n\mathbb{Z}$. We present results on the tightness of the known inequality $\gamma(X_{\mathbb{Z}/n\mathbb{Z}}) \leq \gamma_t(X_{\mathbb{Z}/n\mathbb{Z}}) \leq g(n)$, where γ andγ_t denote the domination number and total domination number, respectively, and g is the arithmetic function known as Jacobsthal's function. In particular, we construct integers n with arbitrarily many distinct prime factors such that $\gamma(X_{\mathbb{Z}/n\mathbb{Z}}) \leq \gamma_t(X_{\mathbb{Z}/n\mathbb{Z}}) \leq g(n) - 1$. We give lower bounds for the domination numbers of direct products of complete graphs and present a conjecture for the exact values of the upper domination numbers of direct products of balanced, complete multipartite graphs.

1. Introduction

If R is a commutative ring with unity, we can define the unitary Cayley graph of R, denoted X_R, as follows. The vertices of X_R are the elements of R and x is adjacent to y if and only if $x - y$ is a unit of R. In this paper we study the domination number and upper domination number of $X_{\mathbb{Z}/n\mathbb{Z}}$. Motivation for studying $X_{\mathbb{Z}/n\mathbb{Z}}$ comes from the theory of graph representation. See Gallian's "Dynamic Survey of Graph Labeling" for more information about the representation numbers of graphs and for additional references [8]. The unitary Cayley graph of $\mathbb{Z}/n\mathbb{Z}$ is highly symmetric and structured, and graph invariants of $X_{\mathbb{Z}/n\mathbb{Z}}$ are well-studied. Often the innate structure of $X_{\mathbb{Z}/n\mathbb{Z}}$ gives rise to pleasing combinatorial results. In 1995 Dejter and Giudici [6] introduced the notion of a unitary Cayley graph and determined the number of triangles in $X_{\mathbb{Z}/n\mathbb{Z}}$. One of the current authors later generalized this result by finding a formula for the number of cliques of any order in $X_{\mathbb{Z}/n\mathbb{Z}}$ [5]. In 2007 Klotz and Sander determined the chromatic number, clique number, independence number, and diameter of $X_{\mathbb{Z}/n\mathbb{Z}}$ [11]. Other properties of unitary Cayley graphs are studied in [1,5,7,14].

It is natural to view unitary Cayley graphs as direct products of balanced, complete multipartite graphs. Throughout this paper let $V(G)$ denote the vertex set of a graph G. If G and H are graphs, then the *direct product* (alternatively called the *tensor product* or *Kronecker product*) of G and H, denoted $G \times H$ (some authors use $G \otimes H$), is defined as follows: $V(G \times H)$ is the Cartesian product $V(G) \times V(H)$, and (g_1, h_1) is adjacent to (g_2, h_2) if and only if g_1 is adjacent to g_2 in G and h_1 is adjacent to h_2 in H. A *balanced, complete k-partite graph* is a graph whose vertices can be partitioned into k different independent sets of equal cardinality such that any two vertices in different independent sets are adjacent. The equal-sized independent sets are called the *partite sets*. We denote by $K[a, b]$ the balanced, complete b-partite graph in which each partite set has size a. Note that $K[1, b]$ is simply the complete graph K_b.

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If p is a prime and α is a positive integer, then it is straightforward to see that $X_{\mathbb{Z}/p^\alpha\mathbb{Z}} \cong K[p^{\alpha-1}, p]$. It follows from the Chinese remainder theorem that if $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the prime factorization of an integer $n > 1$, then $X_{\mathbb{Z}/n\mathbb{Z}} \cong K[p_1^{\alpha_1-1}, p_1] \times \cdots \times K[p_k^{\alpha_k-1}, p_k]$. The authors of [1] have shown more generally that the unitary Cayley graph of any finite commutative ring is a direct product of balanced, complete multipartite graphs. Therefore, we will state many of our results in the more general framework of direct products of balanced, complete multipartite graphs.

This article focuses primarily on two well-studied graph parameters related to dominating sets. We say a vertex u of a graph G *dominates* a vertex v if $u = v$ or u is adjacent to v. A *dominating set* of G is a set $D \subseteq V(G)$ such that every vertex in $V(G)$ is dominated by an element of D. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of G. We call a dominating set D *minimal* if no proper subset of D is a dominating set. The *upper domination number* of G, denoted $\Gamma(G)$, is the maximum size of a minimal dominating set of G. We also find it convenient to define a *total dominating set* of G to be a set $D \subseteq V(G)$ such that every vertex in $V(G)$ is adjacent to an element of D. The minimum cardinality of a total dominating set of G, called the *total domination number* of G, is denoted by $\gamma_t(G)$. Since every total dominating set is a dominating set, we have the trivial inequality $\gamma_t(G) \geq \gamma(G)$. For much more information about domination in graphs, especially in graph products, see [2,9,15,16] and the references therein.

In 2010 Mekiš provided bounds for the domination numbers of certain direct products of complete graphs. We restate some of these results in Theorem 2.1 and devote the rest of that section to developing techniques for proving further bounds. For example, one specific application of our results shows that if $2 = n_1 \leq n_2 \leq n_3 \leq n_4$ and $G = \prod_{i=1}^{4} K_{n_i}$, then $\gamma(G) = 8$ (the product denotes the graph direct product).

Let $g(n)$ denote the smallest positive integer m such that every set of m consecutive integers contains an element that is relatively prime to n. The arithmetic function g is known as Jacobsthal's function; it has received a fair amount of attention from number theorists, partly because of its applications to the study of prime gaps and the study of the smallest primes in arithmetic progressions [10,13,18,19]. In 2013 Maheswari and Manjuri [12] claimed that $\gamma(X_{\mathbb{Z}/n\mathbb{Z}}) = g(n)$ when n has at least 3 distinct prime factors. Their proof correctly shows that $\gamma(X_{\mathbb{Z}/n\mathbb{Z}}) \leq g(n)$. This is simply because $\{0, 1, \ldots, g(n) - 1\}$ is a dominating set of $X_{\mathbb{Z}/n\mathbb{Z}}$. In fact, this set is a total dominating set of $X_{\mathbb{Z}/n\mathbb{Z}}$, so we actually know the stronger inequality $\gamma_t(X_{\mathbb{Z}/n\mathbb{Z}}) \leq g(n)$. However, in 2016 one of the current authors [5] noted that $\gamma(X_{\mathbb{Z}/30\mathbb{Z}}) = 4 < 6 = g(30)$. In general, $\gamma(X_{\mathbb{Z}/n\mathbb{Z}})$ is not necessarily equal to $g(n)$. In Section 3 we provide results that help to quantify when and how drastically the inequality $\gamma(X_{\mathbb{Z}/n\mathbb{Z}}) \leq g(n)$ fails to be an equality. Specifically, we show that for each positive integer j, there is an integer n with more than j distinct prime factors such that $\gamma(X_{\mathbb{Z}/n\mathbb{Z}}) \leq \gamma_t(X_{\mathbb{Z}/n\mathbb{Z}}) < g(n)$.

In Section 4 we conjecture that $\Gamma(X_{\mathbb{Z}/n\mathbb{Z}}) = n/p_1$, where p_1 is the smallest prime factor of n. We prove this conjecture for all n where $p_1 = 2$ and in some additional cases. We state the conjecture and our results in the more general setting of direct products of balanced, complete multipartite graphs.

2. Domination in direct products of complete graphs

In this section, we develop techniques for proving estimates for the domination numbers of direct products of complete graphs that are independent of our focus on unitary Cayley graphs. We generalize a theorem of Mekiš in Theorem 2.6. The only result from this section that will be invoked in subsequent sections is Theorem 2.9, which states that $\gamma(G) = 8$ when G is the direct product of K_2 and three other complete graphs. Therefore, the reader interested only in the subsequent sections may safely pass over the current one.

In [15], Mekiš studied the domination numbers of graphs of the form $\prod_{i=1}^{t} K_{n_i}$, where K_n denotes the complete graph on n vertices (recall that the product denotes the graph direct product). For completeness, we summarize some of his results in the following theorem.

Theorem 2.1 (*Mekiš*). *Let $G = \prod_{i=1}^{t} K_{n_i}$, where $2 \leq n_1 \leq n_2 \leq \cdots \leq n_t$. If $t = 2$, then*

$$\gamma(G) = \begin{cases} 2, & \text{if } n_1 = 2; \\ 3, & \text{if } n_1 \geq 3. \end{cases}$$

If $t = 3$, then $\gamma(G) = 4$. For $t \geq 3$, we have $\gamma(G) \geq t + 1$, and equality holds if $n_1 \geq t + 1$.

Even when considering the domination numbers of more general direct products of balanced, complete multipartite graphs, it is useful to know lower bounds for the domination numbers of direct products of complete graphs. This is because of the following lemma, whose straightforward proof we omit.

Lemma 2.2. *For any positive integers $a_1, a_2, \ldots, a_t, b_1, b_2, \ldots, b_t$, we have*

$$\gamma\left(\prod_{i=1}^{t} K[a_i, b_i]\right) \geq \gamma\left(\prod_{i=1}^{t} K_{b_i}\right).$$

The next lemma builds upon the last line in Theorem 2.1 by giving upper bounds for $\gamma_t(G)$ (hence, also for $\gamma(G)$) under specific conditions on the sizes of n_1 and n_2. Recall that the vertices of the graph $\prod_{i=1}^{t} K_{n_i}$ are t-tuples in which the i^{th} coordinate is a vertex in K_{n_i}. Throughout the rest of this section, we denote the ith coordinate of a vertex x in this direct product by $[x]_i$. Vertices x and y are adjacent if and only if $[x]_i \neq [y]_i$ for all $1 \leq i \leq t$.