Discrete Mathematics 341 (2018) 2753–2761

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Hall's and Kőnig's theorem in graphs and hypergraphs

Isabel Beckenbach *, Ralf Borndörfer

Zuse Institute Berlin, Takustr. 7, 14195 Berlin, Germany

ARTICLE INFO

Article history:
Received 27 November 2017
Received in revised form 29 May 2018
Accepted 3 June 2018

Keywords:
Matching
Hall's theorem
Kőnig's theorem
Normal hypergraph

ABSTRACT

We investigate the relation between Hall's theorem and Kőnig's theorem in graphs and hypergraphs. In particular, we characterize the graphs satisfying a deficiency version of Hall's theorem, thereby showing that this class strictly contains all Kőnig–Egerváry graphs. Furthermore, we give a generalization of Hall's theorem to normal hypergraphs.

1. Introduction

Hall's theorem gives a necessary and sufficient condition for the existence of a system of distinct representatives of a family of finite sets. It is equivalent to the following result on matchings in bipartite graphs.

Theorem 1.1 ([6]). *A bipartite graph has a perfect matching if and only if for all stable sets S the set $N(S)$ of its neighbors is as least as big as S.*

The optimization version of Hall's theorem is Kőnig's theorem.

Theorem 1.2. *In a bipartite graph the maximum size of a matching is equal to the minimum size of a vertex cover.*

The two theorems are equivalent in the sense that one can be proven using the other. In this article we look at both theorems from a different perspective. Namely, we characterize the graphs which satisfy them. In the case of Kőnig's theorem, graphs in which the maximum size of a matching equals the minimum size of a vertex cover are called Kőnig–Egerváry graphs and were characterized by Deming [4] and Sterboul [14]. However, graphs satisfying Hall's theorem have not received much attention. Therefore, we investigate these graphs. Furthermore, we look at variants of Kőnig's and Hall's theorem for hypergraphs.

2. Relations between graph properties

In this section we look at the relationship between Kőnig's theorem, Hall's theorem, and the deficiency version of Hall's theorem in several graph classes including bipartite and Kőnig–Egerváry graphs.

First of all we need some notation. We denote by $\nu(G)$ the maximum size of a matching in a graph G, and by $\nu^*(G)$ the maximum size of a fractional matching. Graphs for which both values are equal are called *stable* graphs. This name comes

* Corresponding author.
E-mail addresses: beckenbach@zib.de (I. Beckenbach), borndoerfer@zib.de (R. Borndörfer).

https://doi.org/10.1016/j.disc.2018.06.013
0012-365X/

from the fact that stable graphs are exactly those graphs for which the set of vertices that are not covered by some maximum matching form a stable set. In particular, we can test whether a graph is stable in polynomial time.

We use the following two notions which can for example be found in [11] and [8]

Definition 2.1 (*Deficiency, Critical Difference of a Graph*)**.** The *deficiency* $\operatorname{def}(G)$ of a graph G is the number of vertices not covered by a maximum matching, i.e., $\operatorname{def}(G) = |V| - 2\nu(G)$. The *critical difference* is defined to be $d(G) := \max\{|S| - |N(S)| : S \text{ is a stable set}\}$.

The deficiency of a graph is as least as large as its critical difference. Namely, if S is a stable set, then every matching has to match a vertex from S to one of $N(S)$. If $|S| \geq |N(S)|$, then every matching leaves at least $|S| - |N(S)|$ vertices uncovered. This implies that the critical difference gives a lower bound on the deficiency.

It is possible that the critical difference is smaller than the deficiency of a graph. For example, the deficiency of K_3 is one where as its critical difference is zero. In particular, every stable set S of K_3 has at least $|S|$ neighbors. Thus, we cannot establish the non-existence of a perfect matching using the condition of Hall's theorem.

In the remainder of this section we characterize the class of graphs for which Hall's theorem holds, i.e., we look at the class containing all graphs G such that G has a perfect matching is equivalent to $|N(S)| \geq |S|$ for all stable sets S of G. Furthermore, we characterize those graphs with $\operatorname{def}(G) = d(G)$. Therefore, we introduce the following two notions.

Definition 2.2 (*(Strong) Hall Property*)**.** A graph G has the *Hall property* if it has a perfect matching or a stable set S with less than $|S|$ neighbors, and it has the *strong Hall property* if $\operatorname{def}(G) = d(G)$.

Hall's theorem states that every bipartite graph has the Hall property. By definition, every graph with the strong Hall property has the Hall property, too.

For graphs the deficiency version of Hall's theorem ($\operatorname{def}(G) = d(G)$) seems to be an easy reformulation of Kőnig's theorem. So one might expect that the graphs with the strong Hall property are just the Kőnig–Egerváry graphs. Indeed, every Kőnig–Egerváry graph has the strong Hall property, see [8]. However, there are other graphs with the strong Hall property. Namely, we show that it is enough that the maximum size of a matching equals the maximum size of a fractional matching.

Theorem 2.1. *A graph has the strong Hall property if and only if it is stable.*

Proof. Let G be any graph. We show that $\operatorname{def}(G) = d(G)$ is equivalent to $\nu(G) = \nu^*(G)$ using integer programming methods and the well-known fact that the vertices of the fractional vertex cover polyhedron are half-integral.

We claim that the critical difference $d(G)$ of G equals the optimal value of the following integer program

$$\max \sum_{v \in V(G)} (b_v - r_v) \tag{1}$$

$$s.t. \sum_{v \in e} (r_v - b_v) \geq 0 \ \forall e \in E(G) \tag{2}$$

$$b_v, r_v \in \{0, 1\} \qquad \forall v \in V(G). \tag{3}$$

Indeed, if S is a stable set, then

$$b_v := \begin{cases} 1, & v \in S \\ 0, & v \notin S, \end{cases} \quad r_v := \begin{cases} 1, & v \in N(S) \\ 0, & v \notin N(S) \end{cases}$$

satisfies the inequalities of type (2)) and its objective value is $|S| - |N(S)|$. Thus, the optimal value of (1)–(3) is at least $d(G)$. On the other hand, let b^*, r^* be an optimal solution to (1)–(3). If there exists a vertex v with $b_v^* = r_v^* = 1$, then decreasing both variables to zero does not affect the feasibility of r^*, b^* and it does not change the objective value. Thus, for every vertex v we can assume that at most one of r_v^*, b_v^* is non-zero. We set $S := \{v | b_v^* = 1\}$ and $T := \{v | r_v^* = 1\}$. Let $\{v, w\}$ be an edge with $v \in S$. Inequality (2) and the modification of r^*, b^* imply that $r_w^* = 1$, i.e., $w \in T$. Thus, S is a stable set and $T \supseteq N(S)$. The objective value of r^*, b^* is equal to $|S| - |T| \leq |S| - |N(S)| \leq d(G)$. In total, we have shown that the critical difference can be calculated via the IP (1)–(3).

Setting $w_v := \frac{r_v - b_v + 1}{2}$ transforms (1)–(3) into the half-integer program

$$\max \left(|V(G)| - 2 \cdot \sum_{v \in V(G)} w_v \right) \tag{4}$$

$$s.t. \sum_{v \in e} w_v \geq 1 \qquad \forall e \in E(G) \tag{5}$$

$$w_v \in \{0, \tfrac{1}{2}, 1\} \qquad \forall v \in V(G). \tag{6}$$