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On the reduced Euler characteristic of independence complexes of circulant graphs

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ABSTRACT

Let G be the circulant graph $C_n(S)$ with $S \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We study the reduced Euler characteristic $\tilde{\chi}$ of the independence complex $\Delta(G)$ for $n = p^k$ with p prime and for $n = 2p^k$ with p odd prime, proving that in both cases $\tilde{\chi}$ does not vanish. We also give an example of circulant graph whose independence complex has $\tilde{\chi}$ which equals 0, giving a negative answer to R. Hoshino.

0. Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. A subset C of $V(G)$ is a *clique* of G if any two different vertices of C are adjacent in G. A subset A of $V(G)$ is called an *independent set* of G if no two vertices of A are adjacent in G. The *complement graph* of G, $\bar{G}$, is the graph with vertex set $V(G)$ and edge set $E(\bar{G}) = \{\{u, v\} \text{ with } u, v \in V(G) \mid \{u, v\} \notin E(G)\}$. In particular, a set is independent in G if and only if it is a clique in the complement graph $\bar{G}$.

We also recall that a circulant graph is defined as follows. Let $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. The *circulant graph* $G := C_n(S)$ is a simple graph with $V(G) = \mathbb{Z}_n = \{0, \ldots, n-1\}$ and $E(G) := \{\{i, j\} \mid |j - i|_n \in S\}$ where $|k|_n = \min\{|k|, n - |k|\}$.

Recently many authors have studied some combinatorial and algebraic properties of circulant graphs (see [7,3,2,12,5,10]). In particular, in [7,3,2,5], a formula for the f-vector of the independence complex was shown for some nice classes of circulants, e.g. the dth power cycle, $S = \{1, 2, \ldots, d\}$, and its complement. Moreover, Hoshino in [7, p. 247] focused on the Euler characteristic, an invariant that is associated to any simplicial complex (see [4]). In particular, he conjectured, by our notation, that any independence complex associated to a non-empty circulant graph has reduced Euler characteristic always different from 0.

We show that for particular n, a circulant graph $C_n(S)$ will support the conjecture, independent of the choice of S. To this aim, we exploit that each entry of the f-vector is a multiple of a divisor of n (see Lemma 2.1).

In Section 2 we prove that the conjecture holds for $n = p^k$ for any prime p, and for $n = 2p^k$ for any odd prime p. Moreover we disprove the conjecture by providing a counterexample (see Example 2.10).

As an application of our results, we focus our attention on two algebraic objects related to the independence complex of circulant graphs. We first consider the *independence polynomial* (see [7,2]), that is

$$I(G, x) = \sum_{i=0}^{n} f_{i-1} x^i, \qquad (0.1)$$

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where f_{i-1} are the entries of the f-vector of the independence complex of a graph G. In particular, -1 is a root of the independence polynomial if and only if the Euler characteristic of the independence complex vanishes. This happens in Example 2.10 and does not happen for all the cases studied in Theorems 2.3, 2.9.

The second application arises from commutative algebra (see e.g. [4,9,14,11]). Let $R = K[x_0, \ldots, x_{n-1}]$ be the polynomial ring and $I(G)$ the edge ideal related to the graph G (see [13]), that is

$$I(G) = (x_i x_j : \{i, j\} \in E(G)). \tag{0.2}$$

In this case the non-vanishing of the reduced Euler characteristic gives us information about the regularity index of $R/I(G)$, namely the smallest integer such that the Hilbert function on $R/I(G)$ becomes a polynomial function, the so-called Hilbert polynomial (see Section 1, Remark 1.2). Also in this case Theorems 2.3, 2.9 and Example 2.10 are relevant.

1. Preliminaries

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article.

Set $V = \{x_1, \ldots, x_n\}$. A *simplicial complex* Δ on the vertex set V is a collection of subsets of V such that: 1) $\{x_i\} \in \Delta$ for all $x_i \in V$; 2) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$. An element $F \in \Delta$ is called a *face* of Δ. A maximal face of Δ with respect to inclusion is called a *facet* of Δ.

The dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of Δ is the maximum of the dimensions of all facets. Let $d-1$ be the dimension of Δ and let f_i be the number of faces of Δ of dimension i with the convention that $f_{-1} = 1$. Then the f-vector of Δ is the $(d+1)$-tuple $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$. The h-vector of Δ is $h(\Delta) = (h_0, h_1, \ldots, h_d)$ with

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}. \tag{1.1}$$

The sum

$$\widetilde{\chi}(\Delta) = \sum_{i=0}^{d} (-1)^{i-1} f_{i-1}$$

is called the *reduced Euler characteristic* of Δ and $h_d = (-1)^{d-1}\widetilde{\chi}(\Delta)$.

Given any simplicial complex Δ on V, we can associate a monomial ideal I_Δ in the polynomial ring R as follows:

$$I_\Delta = (\{x_{j_1} x_{j_2} \cdots x_{j_r} : \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\} \notin \Delta\}).$$

R/I_Δ is called *Stanley–Reisner ring* and its Krull dimension is d. If G is a graph, the *independence complex* of G is

$$\Delta(G) = \{A \subset V(G) : A \text{ is an independent set of } G\}.$$

The independence polynomial is associated to $\Delta(G)$ and by Eq. (0.1) it follows

$$\widetilde{\chi}(\Delta(G)) = -I(G, -1). \tag{1.2}$$

We also remark that from the definition of Stanley–Reisner ring and by Eq. (0.2), it follows $R/I_{\Delta(G)} = R/I(G)$.

The *clique complex* of a graph G is the simplicial complex whose faces are the cliques of G.

Remark 1.1. Let $G = C_n(S)$ be a circulant graph on $S \subseteq T := \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. We observe that the complement graph of G, namely $\bar{G}$, is a circulant graph on $\bar{S} := T \setminus S$. Moreover the clique complex of $\bar{G}$ is the independence complex of G, $\Delta(G)$.

We also recall some basic facts about the regularity index (see [14, Chapter 5]). Let R be a standard graded ring and I be a homogeneous ideal. The *Hilbert function* $\mathrm{H}_{R/I} : \mathbb{N} \to \mathbb{N}$ is defined by

$$\mathrm{H}_{R/I}(k) := \dim_K (R/I)_k$$

where $(R/I)_k$ is the k-degree component of the gradation of R/I (see [13, Section 2.2]), while the Hilbert–Poincaré series of R/I is

$$\mathrm{HP}_{R/I}(t) := \sum_{k \in \mathbb{N}} \mathrm{H}_{R/I}(k) t^k.$$

By the Hilbert–Serre theorem, the Hilbert–Poincaré series of R/I is a rational function, in particular

$$\mathrm{HP}_{R/I}(t) = \frac{h(t)}{(1-t)^n}$$

for some $h(t) \in \mathbb{Z}[t]$. There exists a unique polynomial $P_{R/I}$ such that $\mathrm{H}_{R/I}(k) = P_{R/I}(k)$ for all $k \gg 0$. The minimum integer $k_0 \in \mathbb{N}$ such that $\mathrm{H}_{R/I}(k) = P_{R/I}(k)$ for all $k \geq k_0$ is called *regularity index* and we denote it by $\mathrm{ri}(R/I)$.