Discrete Mathematics 341 (2018) 2774–2788

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Extremal problems on the Hamiltonicity of claw-free graphs

Binlong Li [a], Bo Ning [b,*], Xing Peng [b]

[a] Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaanxi, 710072, PR China
[b] Center for Applied Mathematics, Tianjin University, Tianjin, 300072, PR China

ARTICLE INFO

Article history:
Received 15 May 2017
Received in revised form 13 June 2018
Accepted 15 June 2018

Keywords:
Hamilton cycles
Claw-free graph
Clique number
Claw-free closure
Eigenvalues

ABSTRACT

In 1962, Erdős proved that if a graph G with n vertices satisfies

$$e(G) > \max\left\{\binom{n-k}{2} + k^2, \binom{\lceil (n+1)/2 \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\},$$

where the minimum degree $\delta(G) \geq k$ and $1 \leq k \leq (n-1)/2$, then it is Hamiltonian. For $n \geq 2k+1$, let $E_n^k = K_k \vee (kK_1 + K_{n-2k})$, where "$\vee$" is the "join" operation. One can observe $e(E_n^k) = \binom{n-k}{2} + k^2$ and E_n^k is not Hamiltonian. As E_n^k contains induced claws for $k \geq 2$, a natural question is to characterize all 2-connected claw-free non-Hamiltonian graphs with the largest possible number of edges. We answer this question completely by proving a claw-free analog of Erdős' theorem. Moreover, as byproducts, we establish several tight spectral conditions for a 2-connected claw-free graph to be Hamiltonian. Similar results for the traceability of connected claw-free graphs are also obtained. Our tools include Ryjáček's claw-free closure theory and Brousek's characterization of minimal 2-connected claw-free non-Hamiltonian graphs.

© 2018 Elsevier B.V. All rights reserved.

1. Introduction

Given a graph G, a *Hamilton cycle* of G is a cycle which visits all vertices of G. We will say that G is *Hamiltonian* if it contains a Hamilton cycle. Determining the Hamiltonicity of a graph is a classically difficult problem in graph theory. An old result due to Ore [33] states that every graph with n vertices and more than $\binom{n-1}{2} + 1$ edges is Hamiltonian. Generalizing Ore's theorem by introducing the minimum degree of a graph as a new parameter, Erdős [10] proved the following theorem:

Theorem 1.1 (*Erdős [10]*). *For a graph G with n vertices and $\delta(G) \geq k$ where $1 \leq k \leq (n-1)/2$, if*

$$e(G) > \max\left\{\binom{n-k}{2} + k^2, \binom{\lceil (n+1)/2 \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2\right\}$$

then G is Hamiltonian.

Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is defined as: $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$. Erdős' theorem is tight as shown by the following graph: let

* Corresponding author.
E-mail addresses: libinlong@mail.nwpu.edu.cn (B. Li), bo.ning@tju.edu.cn (B. Ning), x2peng@tju.edu.cn (X. Peng).

https://doi.org/10.1016/j.disc.2018.06.023
0012-365X/© 2018 Elsevier B.V. All rights reserved.

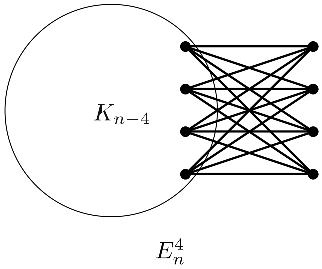

Fig. 1. The graph E_n^4.

$E_n^k := K_k \vee (kK_1 + K_{n-2k})$, where $n \geq 2k+1$ (see Fig. 1 for an example). When $k < n/6$, we have

$$e(E_n^k) = \binom{n-k}{2} + k^2 \geq \binom{\lceil (n+1)/2 \rceil}{2} + \left\lfloor \frac{n-1}{2} \right\rfloor^2.$$

However, E_n^k is not Hamiltonian. We say a graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. We remark that E_n^k is not claw-free for $k \geq 2$ and the condition $\delta(G) \geq 2$ is necessary for a graph to be Hamiltonian.

Claw-free graphs play an important role when we consider the Hamiltonicity of graphs. A long-standing conjecture by Matthews and Sumner [28] asserts that every 4-connected claw-free graph is Hamiltonian. They also constructed 3-connected claw-free graphs which are not Hamiltonian. Therefore, it is natural to consider pairs of forbidden subgraphs which force a 2-connected graph to be Hamiltonian. Bedrossian [1] solved this problem completely by proving that if R and S are connected graphs of order at least 3 with $R, S \neq P_3$ and G is 2-connected, then G being R-free and S-free implies G is Hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$, or W (see [1]). Recently, Bedrossian's result has received a lot of attention. Li et al. [25] and Ning and Zhang [31] obtained heavy subgraph versions of Bedrossian's result by restricting Ore-type degree sum condition [32] and Fan-type 2-distance condition [11] to induced subgraphs, respectively. Very recently, Li and Vrána [26] characterized all disconnected forbidden pairs for a 2-connected graph to be Hamiltonian. In the other direction, Brousek [5] characterized some important properties of minimal 2-connected claw-free non-Hamiltonian graphs, from which Bedrossian's result can be obtained as a corollary.

The main goal of this paper is to give claw-free analogs of Erdős' theorem. We first consider the following problem:

Problem 1. Can we characterize all 2-connected claw-free non-Hamiltonian graphs on n vertices that have the largest number of edges?

We obtain the following solution to Problem 1, and point out that Brousek's result is a key ingredient in our proof.

Theorem 1.2. *Let G be a 2-connected claw-free graph on $n \geq 24$ vertices. If $e(G) \geq e(EB_n) = e(EB'_n)$, then G is Hamiltonian unless $G = EB_n$ or $G = EB'_n$ (see Fig. 2).*

Moreover, we prove a general Erdős-type result for the Hamiltonicity of 2-connected claw-free graphs involving minimum degree and number of edges. We define the graph $F_{k+1,k+1,n-2k-2}$ as: $V(G) = \bigcup_{i=1}^{3} V(G_i)$, where $G_1 = G_2 = K_{k+1}$, $G_3 = K_{n-2k-2}$, and G_1, G_2, G_3 are pairwise vertex-disjoint; $E(G) = \bigcup_{i=1}^{3} E(G_i) \bigcup (\bigcup_{1 \leq i < j \leq 3} E_G(G_i, G_j))$, where $E_G(G_i, G_j) = \{u_i u_j, v_i v_j : 1 \leq i < j \leq 3, u_i, v_i \in V(G_i) \text{ and } u_i \neq v_i, 1 \leq i \leq 3\}$. Obviously, $\delta(F_{k+1,k+1,n-2k-2}) = k$.

Theorem 1.3. [1] *Let $k \geq 3$ and $n \geq k^2 + 8k + 4$. Suppose that G is a 2-connected claw-free graph of order n and minimum degree $\delta(G) \geq k$. If*

$$e(G) \geq e(F_{k+1,k+1,n-2k-2}) = \binom{n-2k-2}{2} + 2\binom{k+1}{2} + 6,$$

then G is Hamiltonian unless $G = F_{k+1,k+1,n-2k-2}$ (see Fig. 3).

Remark 1. By computation, we have $\binom{n-2k-2}{2} + 2\binom{k+1}{2} + 6 < \binom{n-k}{2} + k^2$ when $n > \frac{3k}{2} + \frac{5}{2} + \frac{4}{k+2}$. Since $n \geq k^2 + 8k + 4$, we can see the inequality always holds. Combining Theorems 1.2 and 1.3, we improve the edge condition of Erdős' theorem for the Hamiltonicity of 2-connected claw-free graphs. Moreover, we observe $EB'_n = F_{3,3,n-6}$.

For a graph G, a *Hamilton path* of G is a path which contains all vertices of G. We say that a graph is *traceable* if it contains a Hamilton path. We use EN_n ($n \geq 6$) to denote the graph obtained from K_{n-3} by adding three disjoint pendent edges (see Fig. 4). Similarly, we have the following sufficient condition for the traceability of connected claw-free graphs.

1 This is a solution to a problem originally appeared in the first version of this paper, which was suggested by one referee.

Download English Version:

https://daneshyari.com/en/article/8902862

Download Persian Version:

https://daneshyari.com/article/8902862

Daneshyari.com