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# Enumerations of peaks and valleys on non-decreasing Dyck paths

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ABSTRACT

A *valley* in a Dyck path is a local minimum, and a *peak* is a local maximum. A Dyck path is *non-decreasing* if the $y$-coordinates of the valleys of the path valley form a non-decreasing sequence. In this paper we provide some statistics about peaks and valleys in non-decreasing Dyck paths, such as their total number, the number of low and high valleys, low and high peaks, etc. Our methods include bijective proofs, recursive relations, and the symbolic method for generating functions.



## 1. Introduction

A Dyck path is a path in the first quadrant of the $xy$-plane that starts at the origin, moves only in the North-East and South-East directions and ends on the $x$-axis [4,5], [10, p. 204]). The *valleys* and *peaks* of a Dyck path $P$ are the local minima and the local maxima, respectively, of $P$.

A Dyck path $P$ is *non-decreasing* if the $y$-coordinates of all valleys of the path $P$ form a non-decreasing sequence (see [1,3]). Barcucci et al. [1] introduced the concept of non-decreasing Dyck paths and gave several related statistics. Deutsch [5] gave algebraic enumerations of several aspects of the general Dyck paths. In particular, he counted the number of low peaks, high peaks, low valleys, and high valleys in such paths. In this paper we consider these and other related questions for the family of non-decreasing Dyck paths. The general results are presented using either the symbolic method [7] or recursions, and our tools include bijective proofs as well. We present the results using closed formulas (most of them in terms of Fibonacci numbers), recursive relations, and generating functions.

Amongst other things we give both a closed formula (in terms of Fibonacci numbers) and the generating function for the total number of peaks in non-decreasing Dyck paths of length $2n$. For two natural numbers $k$ and $n$ we give the number of peaks of altitude $k$ that are in all non-decreasing Dyck paths of length $2n$, as well as the peaks in a position $k$. We provide a relation between altitudes of peaks and pyramid weights, show a relationship between the number of peaks, the number of valleys and the number of non-decreasing Dyck paths. We show analogous results for valleys.

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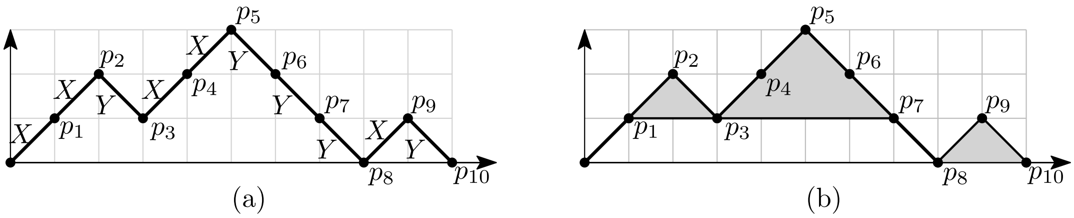


**Fig. 1.** A Dyck path of length 10.

## 2. Preliminaries

The preliminaries given in this section are similar to Czabarka et al. [3]. A *path* $P$ of length $m$ is an $(m+1)$-tuple of points in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that the $i$th point has $i$ as its first coordinate. A *step* in a path $P = (p_0, p_1, \dots, p_m)$ is a pair of two consecutive points $(p_i, p_{i+1})$ for $i \in \{0, \dots, m-1\}$. A *North-East* step has the form $(p_i, p_{i+1}) = ((i, j), (i+1, j+1))$, and a *South-East* step has the form $(p_i, p_{i+1}) = ((i, j), (i+1, j-1))$. The *altitude* of $p_i = (i, j)$, denoted by $alt(p_i)$, is the second coordinate $j$ of $p_i$. We identify a path $P = (p_0, p_1, \dots, p_m)$ with the graph obtained by joining $p_i$ to $p_{i+1}$ with a line segment for $i \in \{0, \dots, m-1\}$.

A *Dyck path* is path $P$ that starts at the origin, is composed of the same number of North-East and South-East steps and has no point with negative altitude. Having the same number of North-East and South-West steps ensures that a Dyck path ends on the $x$-axis and has even length, the nonnegativity of the heights of the points is equivalent with the condition that the graph of the path stays on or above the $x$-axis. We will denote a North-East step by $X$ and a South-East step by $Y$ (see Fig. 1 Part (a)).

A *valley* $v$ of a Dyck path $P = (p_0, p_1, \dots, p_m)$ is a local minimum of the graph of $P$ such that $v \neq p_0, p_m$. Similarly, a *peak* of $P$ is a local maximum of the corresponding graph of $P$. For the Dyck path in Fig. 1 Part (a), $p_3$ and $p_8$ are valleys; and $p_2$, $p_5$ and $p_9$ are peaks.

A *pyramid* $\Delta$ of a Dyck path $P = (p_0, p_1, \dots, p_m)$ is a maximal length tuple of the form

$$(p_{k-r}, \dots, p_r, \dots, p_{k+r})$$

where $r \leq k \leq m-r$ and for each $i : 1 \leq i \leq k$ we have $alt(p_{r-i}) = alt(p_i) - r$ and $alt(p_i + r) = alp(p_i) + r$; i.e. the pyramid is a sequence of $k$ North-East steps followed by a sequence of $k$ South-East steps such that these sequences cannot be extended to a longer sequence in at least one direction. In this case we say that $k$ is the *weight* of the pyramid $\Delta$ see [3] ($\Delta_k$ denotes a pyramid of weight $k$). For example, the weights of first, second and third pyramid in Fig. 1 Part (b) are 1, 2, and 1, respectively.

Let $v_1, \dots, v_t$ be all valley points of a Dyck path $P$ (ordered from left to right). We say that $P$ is *non-decreasing* if

$$alt(v_1) \leq alt(v_2) \leq \cdots \leq alt(v_t).$$

That is, a Dyck path is non-decreasing if the $y$-coordinates of the local minima of the graph form a non-decreasing sequence. The collection of all 13 non-decreasing Dyck paths of length eight is depicted in Table 1. In general, the number of non-decreasing Dyck paths of length $2n$ is the Fibonacci number $F_{2n-1}$ [1,3]. We recall that $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$.

**Theorem 1** (*[1]*). *If $n \in \mathbb{Z}_{\geq 1}$, then the number of non-decreasing Dyck paths of length $2n$ is $F_{2n-1}$.*

Let $\mathcal{D}_n$ be the set of all non-decreasing Dyck paths of length $2n$ and let $d_n = |\mathcal{D}_n|$. The set $\mathcal{D}_n$ can be partitioned into two disjoint sets $\mathcal{A}_n$ and $\mathcal{B}_n$, where $\mathcal{A}_n$ contains those paths that have at least one valley of altitude 0, and $\mathcal{B}_n = \mathcal{D}_n \setminus \mathcal{A}_n$ (see Table 1). We use $\dot\cup$ to mean the disjoint union of sets. Note that

$$\mathcal{D}_n = \mathcal{A}_n \,\dot\cup\, \mathcal{B}_n, \tag{1}$$

and

$$\mathcal{A}_n = \dot{\bigcup_{i=1}^{n-1}} \mathcal{C}_{n,i},$$

where $\mathcal{C}_{n,i}$ contains those paths whose first valley point is at $(2i, 0)$. Thus, we obtain the following disjoint union

$$\mathcal{D}_n = \left( \dot{\bigcup_{i=1}^{n-1}} \mathcal{C}_{n,i} \right) \dot\cup\, \mathcal{B}_n. \tag{2}$$

Removing the path until the first valley point from all paths in $\mathcal{C}_{n,j}$, we can map $\mathcal{C}_{n,j}$ into $\mathcal{D}_{n-j}$ bijectively. Thus, if

$$E_{n-i} = \{P \setminus \Delta_i \mid P \in \mathcal{C}_{n,i} \text{ and } \Delta_i \text{ is the first pyramid in } P\}, \tag{3}$$