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Spanning Euler tours and spanning Euler families in hypergraphs with particular vertex cuts

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ABSTRACT

An *Euler tour* in a hypergraph is a closed walk that traverses each edge of the hypergraph exactly once, while an *Euler family*, first defined by Bahmanian and Šajna, is a family of closed walks that jointly traverse each edge exactly once and cannot be concatenated. In this paper, we study the notions of a *spanning Euler tour* and a *spanning Euler family*, that is, an Euler tour (family) that also traverses each vertex of the hypergraph at least once. We examine necessary and sufficient conditions for a hypergraph to admit a spanning Euler family, most notably when the hypergraph possesses a vertex cut consisting of vertices of degree two. Moreover, we characterise hypergraphs with a vertex cut of cardinality at most two that admit a spanning Euler tour (family). This result enables us to reduce the problem of existence of a spanning Euler tour (which is NP-complete), as well as the problem of a spanning Euler family, to smaller hypergraphs.

1. Introduction

One of the best known and most accessible results in graph theory, Euler's Theorem, states that a connected graph admits an Euler tour – that is, a closed walk traversing each edge of the graph exactly once – if and only if all vertices of the graph have even degree. In addition to the most obvious way to generalise the notion of an Euler tour to hypergraphs, which has been studied in [1,3,5], Bahmanian and Šajna [1,3] also introduced the notion of an *Euler family*, which is a family of closed walks that jointly traverse each edge of the hypergraph exactly once and cannot be concatenated. For connected graphs, the notions of an Euler tour and Euler family coincide; for general connected hypergraphs, however, they give rise to two rather distinct problems, the former NP-complete and the latter of polynomial complexity [3,5].

In this paper, we investigate eulerian substructures that are *spanning*; that is, in addition to traversing each edge exactly once, they also traverse each vertex of the hypergraph at least once. In a connected graph, every Euler tour is spanning; in a general connected hypergraph, however, not every Euler tour or family is spanning.

This paper is organised as follows. After an overview of basic hypergraph terminology in Section 2, we present in Section 3.1 some basic necessary conditions for a hypergraph to admit a spanning Euler family, as well as a characterisation of such hypergraphs via their incidence graphs. In Sections 3.2–3.5, we then focus on the impact of particular vertex cuts on the existence of a spanning Euler tour (family). In our first main result, Theorem 3.7, we show that a hypergraph H with a minimal vertex cut consisting of vertices of degree two admits a spanning Euler family if and only if some of its derived hypergraphs (hypergraphs closely related to particular subhypergraphs of H) admit spanning Euler families. Moreover, in Theorems 3.11 and 3.18–3.21, we show that a hypergraph with a vertex cut of cardinality at most two admits a spanning Euler family (tour) if and only if some of its derived hypergraphs admit spanning Euler families (tours). Hence, when studying the

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problem of existence of a spanning Euler family or tour, it suffices to consider connected hypergraphs without such vertex cuts, thereby reducing the problem.

2. Preliminaries

We begin with some basic concepts related to hypergraphs, which will be used in later discussions. For any graph- and hypergraph-theoretic terms not defined here, we refer the reader to [4] and [2], respectively.

A *hypergraph* H is an ordered pair (V, E), where V is a non-empty finite set and E is a finite multiset of elements of 2^V. (To denote multisets, we shall use double braces, $\{\!\{\cdot\}\!\}$.) The elements of $V = V(H)$ and $E = E(H)$ are called *vertices* and *edges*, respectively. A hypergraph is said to be *empty* if it has no edges.

Let $H = (V, E)$ be a hypergraph, and $u, v \in V$. If $u \neq v$ and there exists an edge $e \in E$ such that $u, v \in e$, then we say that u and v are *adjacent (via the edge e)*, and that u is a *neighbour* of v in H. The set of all neighbours of v in H is called the *neighbourhood* of v in H, and is denoted by $N_H(v)$. Two distinct edges $e, f \in E$ are *adjacent* in H if $e \cap f \neq \emptyset$. If $v \in V$ and $e \in E$ are such that $v \in e$, then v is said to be *incident* with e, and the ordered pair (v, e) is called a *flag* of H. The set of flags of H is denoted by $F(H)$. The *degree* of a vertex $v \in V$ is the number of edges in E incident with v, and is denoted by $\deg_H(v)$, or simply $\deg(v)$ when there is no ambiguity. A vertex of degree 1 is called *pendant*.

The *incidence graph* of a hypergraph $H = (V, E)$ is the graph $\mathcal{G}(H) = (V_G, E_G)$ where $V_G = V \cup E$ and $E_G = \{ve : (v, e) \in F(H)\}$. Hence, $\mathcal{G}(H)$ is simple with bipartition $\{V, E\}$, and E_G can be identified with the flag set $F(H)$. Furthermore, we call $x \in V_G$ a *v-vertex* if $x \in V$, and an *e-vertex* if $x \in E$.

A hypergraph $H' = (V', E')$ is called a *subhypergraph* of the hypergraph $H = (V, E)$ if $V' \subseteq V$ and $E' = \{\!\{e \cap V' : e \in E''\}\!\}$ for some submultiset E'' of E. For any subset $V' \subseteq V$, we define the *subhypergraph of H induced by V'* to be the hypergraph (V', E') with $E' = \{\!\{e \cap V' : e \in E, e \cap V' \neq \emptyset\}\!\}$. Thus, we obtain the subhypergraph induced by V' by deleting all vertices in $V - V'$ from V and from each edge of H, and subsequently deleting all empty edges. By $H \setminus V'$ we denote the subhypergraph of H induced by $V - V'$, and for $v \in V$, we write shortly $H \setminus v$ instead of $H \setminus \{v\}$. Similarly, for any subset $E' \subseteq E$, we denote the subhypergraph $(V, E - E')$ of H by $H - E'$, and for $e \in E$, we write $H - e$ instead of $H - \{e\}$.

A k-length (v_0, v_k)-*walk* in a hypergraph H is an alternating sequence $W = v_0 e_1 v_1 \dots v_{k-1} e_k v_k$ of (possibly repeated) vertices and edges such that $v_0, \dots, v_k \in V$, $e_1, \dots, e_k \in E$, and for each $i \in \{1, \dots, k\}$, the vertices v_{i-1} and v_i are adjacent in H via the edge e_i. Note that since adjacent vertices are by definition distinct, no two consecutive vertices in a walk can be the same. It follows that no walk in a hypergraph may contain an edge of cardinality less than 2. The vertices in $V_a(W) = \{v_0, \dots, v_k\}$ are called the *anchors* of W, v_0 and v_k are the *endpoints* of W, and $v_1, \dots, v_{k-1}$ are the *internal vertices* of W. An *anchor flag* of W is a flag (v, e) such that either ve or ev is a subsequence of W. The multiset of all anchor flags of W is denoted $F(W)$. We also define the edge set of W to be $E(W) = \{e_1, \dots, e_k\}$. Walks W and W' in a hypergraph H are said to be *edge-disjoint* if $E(W) \cap E(W') = \emptyset$, and *anchor-disjoint* if $V_a(W) \cap V_a(W') = \emptyset$.

A walk $W = v_0 e_1 v_1 \dots v_{k-1} e_k v_k$ is called *closed* if $v_0 = v_k$ and $k \geq 2$; a *(strict) trail* if the edges $e_1, \dots, e_k$ are pairwise distinct; a *path* if it is a trail and the vertices $v_0, \dots, v_k$ are pairwise distinct; and a *cycle* if it is a closed trail and the vertices $v_0, \dots, v_{k-1}$ are pairwise distinct. Note that a strict trail as defined above has no repeated anchor flags. In [2], a walk with this property, but possibly with repeated edges, was called a *trail*. In this paper, we shall consider only strict trails, and hence use the shorter term "trail" to mean a "strict trail".

A walk $W = v_0 e_1 v_1 \dots v_{k-1} e_k v_k$ is said to *traverse* a vertex v, an edge e, and an anchor flag (w, f) if $v \in V_a(W)$, $e \in E(W)$, and $(w, f) \in F(W)$, respectively. More precisely, W traverses $e \in E$ exactly t times if $e = e_i$ for exactly t indices $i \in \{1, \dots, k\}$, and traverses $v \in V$ exactly t times if $v = v_i$ for exactly t indices $i \in \{1, \dots, k\}$ in the case W is closed, and exactly t indices $i \in \{0, 1, \dots, k\}$ otherwise.

Vertices u and v are *connected* in a hypergraph H if there exists a (u, v)-walk (equivalently, a (u, v)-path [2, Lemma 3.9]) in H, and H itself is *connected* if every pair of vertices in V are connected in H. The *connected components* of H are the maximal connected subhypergraphs of H without empty edges. The number of connected components of H is denoted by $c(H)$.

An *Euler tour* of a hypergraph H is a closed trail of H traversing every edge of H. An *Euler family* of H is a set of pairwise edge-disjoint and anchor-disjoint closed trails of H jointly traversing every edge of H. In particular, the trails in an Euler family cannot be concatenated.

3. Spanning Euler tours and spanning Euler families

Definition 3.1. An Euler tour T of a hypergraph H is said to be *spanning* if every vertex of H is an anchor of T. An Euler family $\mathcal{F}$ of a hypergraph H is said to be *spanning* if every vertex of H is an anchor of exactly one trail in $\mathcal{F}$.

Since a closed trail T in a hypergraph is a (spanning) Euler tour if and only if $\{T\}$ is a (spanning) Euler family, a (spanning) Euler tour may be thought of as a (spanning) Euler family consisting of a single closed trail. However, a hypergraph may admit a spanning Euler family but no spanning Euler tour (see Fig. 3).

Observe that a hypergraph admits a spanning Euler family if and only if each of its connected components admits a spanning Euler family. Therefore, we may limit our investigation of spanning Euler families to connected hypergraphs, and since empty edges have no impact on connectedness, we shall assume our hypergraphs have no empty edges.